SPECIAL TOPICS IN NUMERICS

(III. EXTRAPOLATION AND DEFEKT CORRECTION)

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6.1 Introduction

The so-called “Richardson[1] extrapolation (to the limit)” and “defect correction (by higher-order differences)” are widely used techniques for increasing the accuracy of low-order discretization methods in solving ordinary as well as partial differential equations. In this chapter, we study its use in solving partial differential equations by the finite element method (FEM).

6.1.1 Concept of ”extrapolation to the limit”

First, we consider an illustrative example with historical origin.

Example 6.1 (Approximation of $\pi = 3.141\dots$): The (transcendental) number π is defined as quotient of the circumference and the diameter of a circle,

$$\pi = \frac{U}{D} = 3,141\,592\,653\,589\,793\,238\,462\,643\,\dots$$

(unit circle: $U = 2\pi$, $D = 2$). In modern Analysis this geometrical description is replaced by the definition of $\frac{1}{2}\pi$ as the (unique) zero of the cosine function in the interval $[0,2]$, what gives the same real number. On the basis of its geometrical definition π can be identified with the circumference of the unit circle determined as the limit of the length of inscribed polygonal courses with increasing fineness.

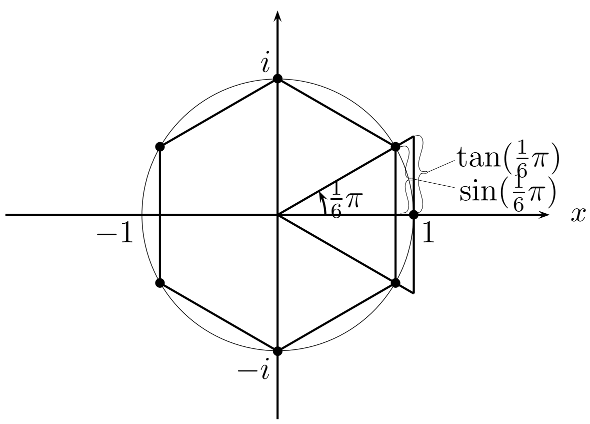

Figure 6.1: Polygonal approximation of the unit circle.

[1]Lewis Fry Richardson (1881–1953): English Mathematician and Physicist; work at various institutions in England and Scotland; typical “applied” mathematician; did pioneering work in modeling and numerics in weather prediction; introduced the concept of extrapolation to the limit in [117].

For the length T_n and U_n of the inscribed half n-gon and its circumscribed counterpart, respectively, there holds

$$T_n = n\,\sin(\pi/n) < \pi < n\,\tan(\pi/n) = U_n.$$

These quantities can be determined recursivly starting from the special values $T_2 = 2\,\sin(\frac{1}{2}\pi) = 2$ and $T_6 = 6\,\sin(\frac{1}{6}\pi) = 3$ by the formula

$$T_{2n} = \sqrt{2n^2 - 2n\sqrt{n^2 - T_n^2}}, \quad n \geq 2.$$

In this way, the Greek Archimedes (287-212 BC) obtaind for $n = 96$ the approximation

$$3,1408\ldots = 3\,\tfrac{10}{71} < \pi < 3\,\tfrac{1}{7} = 3,14285\ldots.$$

By averaging these bounds, Ptolemaios (150 AD) obtained $\pi \approx 3,14166$. Lin Hui (263 AD) obtained for $n = 3072$ the value $\pi \approx 3,14159$ and Al-Kasi (1427) for $n = 3 \cdot 2^{28}$ the value $\pi \approx 3.1415926535897932$, with 17 accurate decimals. Ludolph van Ceulen (1600) computed for $n = 3 \cdot 2^{60}$ an approximation with 35 exact decimals; after him π is also named "Ludolph's number". On the basis of geometrical arguments Huygens[2] (1629-1695) proposed linear combinations

$$S_n := \tfrac{1}{3}\big(4T_n - T_{n/2}\big) \approx \pi,$$

for generating improved approximations. In this way, one obtains for $n = 96$ the improved value $\pi \approx S_{96} = 3,141592$. Behind this approach, we see the basic principle of "extrapolation to the limit". The theoretical explanation for its striking success is that by the above linear combination the leading term $\frac{\pi^3}{3!}h^2$ is eliminated in the following series representation of the sine function:

$$T(h) = \frac{1}{h}\,\sin(h\pi) = \pi - \frac{\pi^3}{3!}h^2 + \frac{\pi^5}{5!}h^4 - \cdots + (-1)^k \frac{\pi^{2k+1}}{(2k+1)!}h^{2k} + \ldots,$$

such that only an error term of higher order $\mathcal{O}(h^4)$ remains.

The general idea of extrapolating a sequence $a(h_i), i \in \mathbb{N}$, of computable values to its (uncomputable) limit $a_0 := \lim_{i\to\infty} a_i$ dates back to the early age of numerical analysis. The idea is quite simple. For a strictly decreasing series of arguments $h_i,\ i = 0,1,\ldots,m$, one computes the values $a(h_i)$ and considers the uniquely determined polynomial interpolating $p_m \in P_m$, $p(h_i) = a(h_i),\ i = 0,1,\ldots,m$. Then, the the value $p(0)$ is taken as an approximation to the desired value a_0, suggesting the name "extrapolation to the limit" (since the limit point $h = 0$ lies outside the interpolation intervall $[h_m, h_0]$). Clearly, this general principle could formally be applied whenever the uncomputable limit

[2]Christiaan Huygens (1629–1695): Dutch Mathematician, Astronomer and Engineer; made important contributions to Analysis and Calculus of Variations; developed the first practically useful teleskope (1654) and the Pendulum Clock (1665).

$a_0 := \lim_{h\to 0} a(h)$ of a continuous computable function $a(h)$ is to be approximated. However, to guarantee that the extrapolated value $p_m(0)$ is actually an improved approximation to a_0 needs a theoretical justification. We will see below that this lies in the existence of a so-called "asymptotic expansion" of the form

$$a(h) = a_0 + \sum_{i=1}^{m} a_i h^{qi} + \mathcal{O}(h_0^{q(m+1)}), \tag{6.1.1}$$

with some factor q, usually $q=1$ or $q=2$. The "hard" part in applying extrapolate to the limit in real-life situations is the proof of such an expansion. The following example is already a bit closer to the subject discussed in this chapter.

Example 6.2 (Numerical differentiation): We consider the numerical differentiation. For C^1-functions f there holds

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x),$$

and for $f \in C^2$,

$$\frac{f(x+h)-f(x)}{h} = f'(x) + \frac{h}{2} f''(\zeta_x), \quad \zeta_x \in \overline{(x, x+h)}.$$

For $f \in C^3$, we have a better approximation to $f'(x)$ by the *central* difference quotient

$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + \frac{h^2}{6} f'''(\zeta_x), \quad \zeta_x \in \overline{(x+h, x-h)}.$$

For analytical f there even holds

$$a(h) := \frac{f(x+h)-f(x-h)}{2h} = f'(x) + \sum_{i=1}^{\infty} \frac{f^{(2i+1)}(x)}{(2i)!} h^{2i},$$

i. e., $a(h)$ is an "even" function in h. In this cade, for the extrapolation of $a(h)$, one uses also "even" polynomials. i. e., polynomials in h^2.

As example, we consider the function $f(x) = \sin(x)$ with $f'(0) = \cos(0) = 1$ and

$$a(h) \equiv \frac{\sin(h) - \sin(-h)}{2h} = \frac{\sin(h)}{h}.$$

Evaluation of $a(h)$ at

$$h_0 = \tfrac{1}{8}, \quad h_1 = \tfrac{1}{16}, \quad h_2 = \tfrac{1}{32} \quad (\text{“goast nodes”: } -\tfrac{1}{8}, -\tfrac{1}{16}, -\tfrac{1}{32})$$
$$a(h_0) = 0.\underline{99}73979, \quad a(h_1) = 0.\underline{999}3491, \quad a(h_2) = 0.\underline{999}8372,$$

then yields

$$p_2(h) = a(h_0)\, \frac{(h^2 - \frac{1}{16^2})(h^2 - \frac{1}{32^2})}{(\frac{1}{8^2} - \frac{1}{16^2})(\frac{1}{8^2} - \frac{1}{32^2})} + \dots, \quad p_2(0) = 0.\underline{9999999}26.$$

Asymptotic error expansions of the form (6.1.1) are well-known in numerical integration (e. g., “Euler-MacLaurin formula” for the composite trapezoidal rule) and in the discretization of ordinary differential equations (ODE) (e. g., Gragg’s expansion of the averaged midpoint rule. Algorithms based on the extrapolation principle are widely used in practice. We refer to the theoretical work of Stetter [128], the survey paper of Joyce [77], and to the monographies of Wasov [136] and of Marchuk & Shaidurov [93] for a comprehensive discussion of extrapolation methods for finite *difference* discretizations. In the context of partial differential equations (PDE), however, the practical use of algorithms directly based on extrapolation is limited to only a few model situations. Apparently, Wasov [135] was the first who rigorously proved an error expansion of third order for the difference discretization of the Poisson equation on a general domain (Shortley-Weller scheme). The special case of a rectangular domain was studied by, e. g., Volkov [131] and Hofmann [75]. Later on these results were extended by Volkov [132] and Pereyra et al. [99] to higher-order interpolatory boundary approximations yielding, qt least in principle, extrapolation schemes of arbitrarily high order. Munz [96] and Böhmer [52] proved error expansions for second-order elliptic operators with variable coefficients. However, all these results rely on the use of a discrete maximum principle in order to establish the stability of the underlying low-order discretization. Therefore, their applicability is restricted to scalar second-order problems.

Most proofs of error expansions for finite difference schemes have the following pattern: A boundary value problem $Lu = f$ is discretized resulting in discrete problems $L_h u_h = r_h f$, where h is a parameter describing the mesh size, and r_h is some “restriction operator” mapping continuous functions into mesh functions. Suppose that the following conditions are satisfied:

a) Regularity: Lu “smooth” implies u “smooth”.

b) Konsistency: $r_h Lu - L_h r_h u = h^q r_h a + o(h^q)$ for “smooth” u .

c) Stability: $\|L_h^{-1}\| \le c$.

Let e be the solution of the auxiliary problem $Le = a$. From (a), we get that e is “smooth”. Then, from (b), for the function $v_h := u_h - r_h u - h^q r_h e$ there holds

$$L_h v_h = r_h Lu - L_h r_h u - h^q L_h r_h e = h^q (r_h a - L_h r_h e) + o(h^q) = o(h^q).$$

Consequently, by (c) it follows that

$$u_h = r_h u + h^q r_h e + o(h^q).$$

The assumption (b) on the truncation error requires the discretization to use a uniform mesh. This, of course, severely limits the flexibility in employing the extrapolation technique for more complicated problems. Its use is even more questionable whem the stability estimate (c) can be guaranteed only in terms of unfavorably strong norms as it is usually the case for finite difference discretizations. Some of these limitations can be overcome or, at least, be weakened by taking a simple finite element Galerkin method as the base scheme for the extrapolation process. In finite element methods sharp stabilit estimates are available and, what is most important, these estimates can be derived without referring to a maximum principle. This allows for significantly more flexibility in designing the discretization, as the consistance condition (b) is needed only with respect to relatively weak norms. Further, the presence of error expansions can now be rigorously proven for situations in which the usual maximum principle does not hold, e. g., for higher-order problems ("plate equation") and for the elliptic systems in linear elasdticity ("Lamé-Navier equations"). Pioneering work in the analysis of extrapolation methods for finite elements was done mainly by Q. Lin and his collaborators [84], [80], [83], and [89] for the two-dimensional Poisson equation. Subsequently, their techniques and results were sharpened and extended in several respects in [48], [85], [103], and [87].

This chapter is very much based on the material developed in the above references, however, using a slightly different presentation in order to stress the analogy to the finite difference discretization. Asymptotic error expansions are closely related to the phenomenon of "superconvergence" in the finite element method. There is an abundant literature on the latter topic dealing with various ways of identifying so-called "stress points" and of improving the accuracy by use of certain averaging procedures. However, this aspect of finite element analysis will not be discussed here. Instead, we refer to the the monography [133] and for a survey and a rather complete list of references on "superconvergence" results in the finite element method.

6.1.2 Concept of "defect correction"

One of the first instances in Numerics when one encounters the concept of "defect correction" is the imprement of accuracy in solving linear algebraic systems by Gaussian[3] elimination subject to round-off error by so-called "residual correction" ("Nachiteration").

The Gaussian elimination algorithm (without "pivoting") transforms a linear system $Ax = b$ into an upper triangular system $Ux = c$, from which the solution x can be obtained by simple backward substitution. This is equivalent to the determination of an LU- decomposition $A = LU$ and the subsequent solution of the two triangular systems

$$Ly = b\,, \quad Ux = y\,. \tag{6.1.2}$$

This variant of the Gaussian algorithm is preferable if the same linear system is to be solved successively for several right-hand sides b. Because of the unavoidable round-off error one usually obtains an only approximate LU-decomposition

$$\tilde{L}\tilde{U} \neq A.$$

and using this in (6.1.2) an only approximate solution $x^{(0)}$ with (exact) "defect" (negative

[3]Carl Friedrich Gauß (1777–1855): eminent German mathematician, astronomer and physicist; worked in Göttingen; fundamental contributions to arithmetic, algebra and geometry; founder of modern number theory, determined the planetary orbits by his "equalization calculus", further contributions to earth magnetism and construction of an electro-magnetic telegraph.

"residual")

$$d^{(0)} := Ax^{(0)} - b \neq 0\,.$$

Of course, this defect is generally also obtaind only approximately as $\hat{d}^{(0)}$ due to round-off errors. Using the already computed approximate trianguler decomposition $\tilde{L}\tilde{R} \sim A$, one solves (again approximately) the so-called "defect equation"

$$Ak = -\hat{d}^{(0)}, \qquad \tilde{L}\tilde{U}k^{(1)} = -\hat{d}^{(0)}, \tag{6.1.3}$$

and from this obtains a correction $k^{(1)}$ for $x^{(0)}$:

$$x^{(1)} := x^{(0)} + k^{(1)}\,. \tag{6.1.4}$$

Had the defect $\hat{d}^{(0)}$ and the defect and the corresponding correction $k^{(1)}$ be computed exactly, i- e., $\hat{d}^{(0)} = d^{(0)}$ and $k^{(1)} = k$, then

$$Ax^{(1)} = Ax^{(0)} + Ak = Ax^{(0)} - d^{(0)} = Ax^{(0)} - Ax^{(0)} + b = b,$$

i. e., $x^{(1)} = x$ would be the exact solution of the system $Ax = b$. In general, $x^{(1)}$ is a better approximation to x than $x^{(0)}$ even if the defect equation is solved only approximately. This, however, requires the computation of the defect $d^{(0)}$ with *higher* accuracy by using extended floating point arithmetic. This is supported by the following error analysis.

We suppose the relative error in computing the LU-decomposition of the matrix A to be bounded by a small number $\varepsilon \ll \mathrm{cond}(A)^{-1}$. Then, by the general perturbation result for linear systems there holds the estimate

$$\frac{\|x^{(0)} - x\|}{\|x\|} \le \frac{\mathrm{cond}(A)}{1 - \mathrm{cond}(A)\frac{\|A-\tilde{L}\tilde{U}\|}{\|A\|}}\ \underbrace{\frac{\|A - \tilde{L}\tilde{U}\|}{\|A\|}}_{\sim\,\varepsilon}\,.$$

Here, the loss of exact decimals corresponds to the condition number $\mathrm{cond}(A)$. Additionally round-off errors are neglected. The exact defect $d^{(0)} := Ax^{(0)} - b$ is replaced by the expression $\hat{d}^{(0)} := \hat{A}x^{(0)} - b$, where $\hat{A}$ is an approximation such that

$$\frac{\|d^{(0)} - \hat{d}^{(0)}\|}{\|A\|\|x^{(0)}\|} = \frac{\|(A - \hat{A})x^{(0)}\|}{\|A\|\|x^{(0)}\|} \le \hat{\varepsilon} \ll \varepsilon\,.$$

By construction there holds

$$\begin{aligned} x^{(1)} = x^{(0)} + k^{(1)} &= x^{(0)} + (\tilde{L}\tilde{U})^{-1}[b - \hat{A}x^{(0)}] \\ &= x^{(0)} + (\tilde{L}\tilde{U})^{-1}[Ax - Ax^{(0)} + (A - \hat{A})\,x^{(0)}], \end{aligned}$$

and, consequently,

$$\begin{aligned} x^{(1)} - x &= x^{(0)} - x - (\tilde{L}\tilde{U})^{-1}A(x^{(0)} - x) + (\tilde{L}\tilde{U})^{-1}(A - \hat{A})\,x^{(0)} \\ &= (\tilde{L}\tilde{U})^{-1}[\tilde{L}\tilde{U} - A](x^{(0)} - x) + (\tilde{L}\tilde{U})^{-1}(A - \hat{A})\,x^{(0)}. \end{aligned}$$

Since

$$\tilde L\tilde U = A - A + \tilde L\tilde U = A\big(I - A^{-1}(A-\tilde L\tilde U)\big),$$

we can use a standard result from matrix perturbation theory to conclude

$$\begin{aligned}\|(\tilde L\tilde U)^{-1}\| &\le \|A^{-1}\|\,\|\,[I - A^{-1}(A-\tilde L\tilde U)]^{-1}\| \\ &\le \frac{\|A^{-1}\|}{1-\|A^{-1}(A-\tilde L\tilde U)\|} \le \frac{\|A^{-1}\|}{1-\mathrm{cond}(A)\frac{\|A-\tilde L\tilde U\|}{\|A\|}} \sim \frac{\mathrm{cond}(A)}{\|A\|}.\end{aligned}$$

This eventually implies

$$\frac{\|x^{(1)}-x\|}{\|x\|} \sim \mathrm{cond}(A)\Big[\underbrace{\frac{\|A-\tilde L\tilde U\|}{\|A\|}}_{\sim\varepsilon}\underbrace{\frac{\|x^{(0)}-x\|}{\|x\|}}_{\sim\mathrm{cond}(A)\varepsilon} + \underbrace{\frac{\|A-\tilde A\|}{\|A\|}}_{\sim\hat\varepsilon}\frac{\|x^{(0)}\|}{\|x\|}\Big].$$

This correction procedure can be iterated to a sucessive "residual correction". It may be continued until the obtained solution has an error (usually achieved after $2-3$ steps) of the order of the defect computation, i. e., $\|x^{(3)}-x\|/\|x\| \sim \hat\varepsilon$.

Example 6.3: The linear system

$$\begin{bmatrix} 1.05 & 1,02 \\ 1.04 & 1,02 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

has the exact solution $x = (-100,\, 103.921\ldots)^T$. Gaussian elimination, with 3-decimal arithmetic and correct rounding, yields the approximate triangular matrices

$$\tilde L = \begin{bmatrix} 1 & 0 \\ 0.990 & 1 \end{bmatrix}, \qquad \tilde U = \begin{bmatrix} 1.05 & 1.02 \\ 0 & 0.01 \end{bmatrix},$$

$$\tilde L\tilde U - A = \begin{bmatrix} 0 & 0 \\ 5\cdot 10^{-4} & 2\cdot 10^{-4} \end{bmatrix} \qquad \text{(correct within machine accuracy).}$$

The resulting "solution" $x^{(0)} = (-97, 1.101)^T$ has the defect

$$d^{(0)} = b - Ax^{(0)} = \begin{cases} (0,0)^T & \text{3-decimal computation,} \\ (0,065, 0,035)^T & \text{6-decimal computation.} \end{cases}$$

The approximate correction equation

$$\begin{bmatrix} 1 & 0 \\ 0.990 & 1 \end{bmatrix}\begin{bmatrix} 1.05 & 1.02 \\ 0 & 0.01 \end{bmatrix}\begin{bmatrix} k_1^{(1)} \\ k_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0.065 \\ 0.035 \end{bmatrix}$$

has the solution $k^{(1)} = (-2.9, 102.899)^T$ (obtained by 3 decimal computation). Hence, one correction step yields the approximate solution

$$x^{(1)} = x^{(0)} + k^{(1)} = (-99.9, 104)^T,$$

which is significantly more accurate than the first approximation $x^{(0)}$.

Another application of the concept of sucessive "defect correction" is in the iterative solution of linear (or nonlinear) algebraic systems obtained systems obtained by the discretozation of an (elliptic) differential equation

$$Lu = f, \tag{6.1.5}$$

resulting in an algebraic system

$$L_h u_h = r_h f,$$

where $h \in \mathbb{R}_+$ indicates a discretization parameter (e. g., mesh width, subspace dimension, etc.) and r_h denotes a certain restriction operator. Then, the often difficult inversion of the discrete operator L_h is "preconditioned" by a "nearby" operator $\tilde{L}_h$ having a simpler structure than L_h and better solution properties. This leads to an iteration process of the form

$$\tilde{L}_h u_h^{t+1} = \tilde{L}_h u_h^t - \big(L_h u_h^t - r_h f\big), \quad t = 1, 2, \dots,$$

starting with some u_h^0, where $d_h^t := L_h u_h^t - r_h f$ is called the "defect" of the approximation u_h^t. This is usually written in form of a "defect correction iteration",

$$\tilde{L}_h \delta u_h^t = -d_h^t, \quad u_h^{t+1} := u_h^t + \delta u_h^t, \quad t = 1, 2, \dots,$$

which saves the computation of the term $\tilde{L}_h u_h^t$. In this iteration the error $e_h^t := u_h - u_h^t$ propagates like

$$e_h^t = \big(I - \tilde{L}_h^{-1} L_h\big) e_h^{t-1}.$$

Hence, if in some operator norm there holds

$$\|I - \tilde{L}_h^{-1} L_h\| =: q < 1,$$

we have convergence

$$\|u_h^t - u_h\| \le q^t \|u_h^0 - u_h\| \to 0 \quad (t \to \infty).$$

Many iterative solvers in Linear Algebra can be casted in this form, e. g., the classical Jacobi and Gauss-Seidel iterations, but also the highly efficient multigrid methods). The same principle can also be used for nonlinear equations

$$L_h(u_h) = r_h f,$$

leading to iterations of the form

$$C_h \delta u_h^t = -d_h^t := r_h f - L_h(u_h^{t-1}), \quad u_{t+1} = u_h^t + \lambda_t \delta u_t, \quad t \ge 1,$$

with appropriately chosen (linear) preconditioning operators C_h and damping parameters $\lambda_t \in (0,1]$. The most famous representative of this class of methods is the classical Newton method where C_h changes in the course of the iteration according to

$$C_h := L'_h(u_h^t).$$

Here, convergence (even quadratic) can be guaranteed for sufficiently good starting values u_h^0 under certain regularity conditions on $L_h(\cdot)$.

Another application in this context, which is closer to the subject of this chapter, is that of accuracy improvement by "higher-order correction". Here, the situation is as follows. For the discretization of the (linear) differential equation (6.1.5) consider a robust but only low-order (order $p > 0$) "base scheme"

$$L_h u_h = r_h f.$$

Suppose that there is another scheme

$$\hat{L}_h \hat{u}_h = \hat{r}_h f,$$

which is formally of higher-order (consistent) in the sense

$$\hat{L}_h r_h u - \hat{r}_h f = o(h^p),$$

but possibly unstable or difficult to solve. Then, in order to increase the accuracy of the "base solution" $u_h =: u_h^{(0)}$, we may use the standard defect correction approach:

$$d_h := \hat{L}_h u_h - \hat{r}_h f, \quad L_h \delta u_h = -d_h, \quad u_h^{(1)} := u_h + \delta u_h.$$

Iterating this process, one gets (exercise)

$$\|u_h^t - r_h u\| \le q_h^t \|u_h - r_h u\| + \frac{q_h}{1-q_h} \|L_h^{-1}\| \|\hat{L}_h r_h u - \hat{r}_h f\|, \quad t \ge 1,$$

where $q_h := \|I - L_h^{-1}\hat{L}_h\|$. If $q_h \ll 1$, only a few corretion steps are needed until the full accuracy of the higher-order scheme is achived. However, normally the theoretical justification of this approach is not based on the contraction property, which may not even hold true, but rather on the existence of an asymptotic error expansion for the base scheme of the form

$$u_h = r_h u + h^p r_h e(u) + o(h^p). \tag{6.1.6}$$

Here, the coefficient $e(u)$ is independent of h and the remainder term is bounded in terms of some higher-order norm of u. In this case, there holds

$$\begin{aligned} L_h(u_h^{(1)} - r_h u) &= L_h\big(u_h - L_h^{-1}(\hat{L}_h u_h - \hat{r}_h f) - r_h u\big) \\ &= (L_h - \hat{L}_h)u_h + \hat{r}_h f - L_h r_h u \\ &= (L_h - \hat{L}_h)\big(r_h u + h^p r_h e(u) + o(h^p)\big) + \hat{r}_h f - L_h r_h u \\ &= h^p (L_h - \hat{L}_h)\big(r_h e(u) + o(1)\big) + \hat{r}_h f - \hat{L}_h r_h u = o(h^p), \end{aligned}$$

from which one infers that

$$u_h^{(1)} = \hat{r}_h u + o(h^p),$$

provided that the base scheme is stable, $\|L_h^{-1}\| \le c$. The main advantage of the just described method consist in its low computational complexity as it requires only the inversion of the base operator L_h. Furthermore, in some applicationa it is important that the higher-order appraximation $\hat{L}_h$ itself does not need to be stable for $h \to 0$. Below, we will discuss the theoretical justification of this kind of defect correction as well as its application in the solution of elliptic and parabolic differential equations by the finite element Galerkin method. For a general analysis of this and related methods with particular emphasis on ordinary differential equations, we refer to the research papers [129], [78], [127], [70], and to the survey article [53].

Asymptotic error expansions of the type (6.1.6) are the central tool in the analysis of extrapolation in finite difference ans finite element discretizations and in the so-called defect correction by "higher-order differences". The classical proofs of asymptotic error expansions for finite difference schemes are based on maximum principles to hold for the base operator L_h, which restricts this approach essentially to second-order scalar problems (see [129] and [93]). This limitation can be overcome in the context of the finite element Galerkin method. Here, the presence of asymptotic error expansions can be rigorously proven even in situations in which the usual maximum principles do not hold, e. g., for the systems and higher-order quations in structural and fluid mechanics. Pioneering work in this direction has been done by Lin Qun and his collaborators in [84, 90]. Their approach has then been developed further in [48], [103], and especially in [46], in order to cover a wider range of problems. These results are the main subject treated in this chapter.

6.2 General theory of extrapolation

6.2.1 Abstract theoretical foundation and practical realization

Let some numerical process deliver for each value of the positive parameter $h \in \mathbb{R}_+$ $(h \to 0)$ an output $a(h) \in \mathbb{R}$. We want to determine the limit value for $h \to 0$, provided of course that it exists, which is not directly computable,

$$a_0 := \lim_{h \to 0} a(h). \tag{6.2.7}$$

For approximating a_0, we compute $a(h_i)$ for certain values $h_0 > h_1 > \ldots > h_n > 0$, and take the value $p_n(0)$ of the (unique) n-degree interpolation polynomial corresponding to the points $(h_i, a(h_i))$ as approximation to a_0. The following theorem provides the theoretical basis for the success of this "extrapolation process".

Theorem 6.1 (Extrapolation error): *Let the function $a(h)$, $h \in \mathbb{R}_+$, possess an asymptozic expansion of the following form:*

$$a(h) = a_0 + \sum_{j=1}^{n} a_j h^{jq} + a_{n+1}(h) h^{(n+1)q} \tag{6.2.8}$$

with some parameter $q > 0$, and certain h-independent coefficients a_j and $a_{n+1}(h) = a_{n+1} + o(1)$, for $h \to 0$. Further, let the sequence $(h_i)_{i=0,1,2,\dots}$ be monotonically decreasing such that

$$0 < \sup_{i\geq 0} \frac{h_{i+1}}{h_i} =: \rho < 1\,. \tag{6.2.9}$$

The, for the interpolation polynomial $p_n^{(i)} \in P_n$ (in h^q) corresponding to the $n+1$ points $(h_i^q, a(h_i))$, $i = 0, \dots, n$, there holds:

$$a(0) - p_n^{(i)}(0) = O(h_i^{(n+1)q}) \quad (i \to \infty). \tag{6.2.10}$$

The constant in this error representation deteriorates for $\rho \to 1$.

Proof. For abbreviation, we set $z := h^q$ and $z_k := h_k^q$. The interpolation polynomial corresponding to the points $(z_{k+i}; a(h_{k+i}))$, $i = 0, \dots, n$, is in Lagrangian[4] form given by

$$p_n(z) = \sum_{i=0}^{n} a(h_{k+i}) L_{k+i}^{(n)}(z)\,, \quad L_{k+i}^{(n)}(z) = \prod_{l=0,\, l\neq i}^{n} \frac{z - z_{k+l}}{z_{k+i} - z_{k+l}}\,.$$

From the error representation of Lagrange interpolation,

$$f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\zeta_x) \prod_{i=0}^{n} (x - x_{k+i})\,, \quad \zeta_x \in [0, h_0]\,,$$

for $f \equiv 1$ and $f(x) = x^r$, we deduce that (exercise)

$$\sum_{i=0}^{n} z_{k+i}^r L_{k+i}^{(n)}(0) = \begin{cases} 1 & , \text{ f"ur } r = 0\,, \\ 0 & , \text{ f"ur } r = 1, \dots, n\,, \\ (-1)^n \prod_{i=0}^{n} z_{k+i} & , \text{ f"ur } r = n+1\,. \end{cases}$$

Using this, we conclude:

[4] *Joseph Louis de Lagrange (1736–1813): French Mathematician; 1766-87 direktor of the Mathem. Class of Berlin Akademy, than Prof. in Paris; fundamental work in Calculus of Variations, Complex Function Theory, and Theoretical and Celestial Mechanics.*

$$p_n(0) = \sum_{i=0}^{n} \Big\{ a_0 + \sum_{j=1}^{n} a_j z_{k+i}^j + a_{n+1}(h_{k+i}) z_{k+i}^{n+1} \Big\} L_{k+i}^{(n)}(0)$$
$$= a_0 \sum_{i=0}^{n} L_{k+i}^{(n)}(0) + \sum_{j=1}^{n} a_j \Big\{ \sum_{i=0}^{n} z_{k+i}^j L_{k+i}^{(n)}(0) \Big\} +$$
$$a_{n+1} \sum_{i=0}^{n} z_{k+i}^{n+1} L_{k+i}^{(n)}(0) + \sum_{i=0}^{n} o(1) z_{k+i}^{n+1} L_{k+i}^{(n)}(0),$$

and, consequently,

$$p_n(0) = a_0 + a_{n+1}(-1)^n \prod_{i=0}^{n} h_{k+i}^q + o(h_k^{(n+1)q}).$$

Here. we have used (6.2.9) to guarantee the h_k-independent estimate

$$|L_{k+i}^{(n)}(0)| = \prod_{l=0, l\neq i}^{n} \left| \frac{z_{k+l}}{z_{k+i} - z_{k+l}} \right| = \prod_{l=0, l\neq i}^{n} \left| \frac{1}{z_{k+i}/z_{k+l} - 1} \right| \leq \gamma(n, \rho) \qquad (6.2.11)$$

Clearly, $|L_{k+i}^{(n)}(0)| \to \infty$ for $\rho \to 1$. Observing

$$\prod_{i=0}^{n} h_{k+i}^q = O(h_k^{(n+1)q})$$

yields (6.2.10) uniformly for all k. Q.E.D.

6.2.2 Practical realization

Usually the extrapolation process is organized in form of the following extrapolation scheme“ (“Extrapolationstableau”) for calculating the values $p_{i,i+k} := p_{i,i+k}(0)$. Recall that $p_{i,i+k}(h)$ is the polynomial (in h^q) which interpolates the $k{+}1$ points $(h_i^q, a(h_i)), \ldots,$ $(h_{i+k}^q, a(h_{i+k}))$. The function values $p_{i,i+k}$ are then obtained by the Neville[5] algorithm:

$$p_{i,i+k} = p_{i,i+k-1} + \frac{p_{i,i+k-1} - p_{i+1,i+k}}{x_{i+k}/x_i - 1}.$$

Following the convention, we set $a_{ik} \equiv p_{i-k,i}$:

[5] *Eric Harold Neville (1889–1961): English Mathematician; Prof. at Univ. of Reading, England (1919–1954); contributions to Numerical Analysis, e. g., practical polynomial interpolation.*

	$a_{i0}=a(h_i)$				
h_0	a_{00}		Extrapolation Scheme		
h_1	a_{10}	$\to a_{11}$			
h_2	a_{20}	$\to a_{21}$	$\to a_{22}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
h_i	a_{i0}	a_{i1}	a_{i2}	$\cdots$	a_{ii}

The table entries are sucessively computed by the following recursion formula:

$$\begin{aligned} i=0,1,2,\dots: \quad & a_{i,0}=a(h_i)\,, \\ i=1,2,3,\dots,\ k=1,2,3,\dots,i: \quad & a_{ik}=a_{i,k-1}+\frac{a_{i,k-1}-a_{i-1,k-1}}{(h_{i-k}/h_i)^q-1}. \end{aligned} \tag{6.2.12}$$

By Theorem 6.1 there holds, for fixed k:

$$a(0)-a_{ik}=O(h_{i-k}^{(k+1)q}) \quad (i\to\infty), \tag{6.2.13}$$

provided that the condition (6.2.9) is met. Possible sequences are $h_i\equiv h_0/n_i$ with

i) $n_i=2^i$, *ii)* $n_i=2,4,6,8,12,16,\dots$, *iii)* $n_i=1,2,3,\dots$ (not admissible).

We note that the step-size condition (6.2.9) is not so restrictive in practice when only a few extrapolation steps are preformed what is usual in the context of solving PDE.

6.2.3 A posteriori error control

In the practical realization of extrapolation to higher order, we need a stopping criterion. Let an error tolerance TOL be given. From the error representation (proof of Theorem 6.1)

$$a_{ik}=a(0)+a_{k+1}(-1)^k\prod_{j=0}^{k}h_{i-k+j}^q+o(h_{i-k}^{(k+1)q}),$$

we conclude that for fixed k and sufficiently large i the error $a_{ik}-a(0)$ converges monotonically to zero (provided that $a_{k+1}\neq 0$). Therefore, with the quantities

$$b_{ik}\equiv 2a_{i+1,k}-a_{ik}$$

there holds case that $q\geq 1$:

$$\begin{aligned} b_{ik}-a(0) &= 2\{a_{i+1,k}-a(0)\}-\{a_{ik}-a(0)\} \\ &= 2a_{k+1}(-1)^k\prod_{j=0}^{k}h_{i+1-k+j}^q+o(h_{i+1-k}^{(k+1)q})-a_{k+1}(-1)^k\prod_{j=0}^{k}h_{i-k+j}^q+o(h_{i-k}^{(k+1)q}) \\ &= \prod_{j=0}^{k}h_{i-k+j}^q\,\{-a_{k+1}(-1)^k+2\,(\frac{h_{i+1}}{h_{i-k}})^q\,a_{k+1}(-1)^k\}+o(h_{i-k}^{(k+1)q}). \end{aligned}$$

Since $h_{i+1}^q/h_{i-k}^q \ll 1$, in first approximation there holds

$$b_{ik} - a(0) \doteq -(-1)^k a_{k+1} \prod_{j=0}^{k} h_{i-k+j}^q,$$

and, consequently,

$$a_{ik} - a(0) \doteq -(b_{ik} - a(0)), \tag{6.2.14}$$

for fixed k and sufficiently large i. Because of the monotone convergence $a_{ik} - a(0) \to 0$ $(i \to \infty)$ it follows that asymptically either $a_{ik} \le a(0) \le b_{ik}$ or $a_{ik} \ge a(0) \ge b_{ik}$, and both sides converge monotonically to $a(0)$ for $i \to \infty$. This behavior of the sequences $(a_{ik})_{i\in\mathbb{N}}$ and $(b_{ik})_{i\in\mathbb{N}}$ (for fixed k) can be used for designing a stopping criterion for the extrapolation process:

$$|a_{ik} - b_{ik}| < \mathrm{TOL} \quad \Rightarrow \quad \text{stop}. \tag{6.2.15}$$

Remark 6.1: In practice is makes sense to carry out the extrapolation process completely, i. e., to compute all diagonal table entries a_{ii}. On can show that the diagonal sequence $(a_{ii})_{i\in\mathbb{N}}$ converges faster to a_0 than any of the column sequences $(a_{ik})_{i\in\mathbb{N}}$, $k \ge 0$; in fact there holds $a_{ii} - a(0) = o(h_{i-k}^{(k+1)q})$. However, this aspect is not of relevance for the use of extrapolation in the PDE context since here usually only a small number of extrapolation steps is carried out.

6.2.4 Application in numerical integration

The composite quadrature formulas with step size $h = \frac{b-a}{N}$ suggest the use of "extrapolation to the limit $h = 0$". In this process the quadrature rules have to be frequently evaluated Therefore, the base formula used in the extrapolation should have a simple structure with as few as possible function evaluations. The standard method of this kind using the composite trapezoidal rule and goes back to Romberg[6] (1955) and carries his name. We set $h = (b-a)/N$ and $x_j = a + jh$, $j = 0, \dots, N$. For the composite trapezoidal rule there holds:

$$\int_a^b f(x)\,dx \le h\left\{\tfrac{1}{2}f(a) + \sum_{j=1}^{N-1} f(x_j) + \tfrac{1}{2}f(b)\right\} - h^2\,\tfrac{b-a}{12}f''(\zeta). \tag{6.2.16}$$

For $f \in C[a,b]$, we have convergence:

$$a(h) = h\sum_{j=0}^{N-1} f(x_j) + \underbrace{\tfrac{1}{2}h\{f(b) - f(a)\}}_{\to 0} \to \int_a^b f(x)\,dx \quad (h \to 0).$$

[6]Werner Romberg (1909–2003): German Mathematician; emigrated 1937 for political reasons to Russia and later to Norway; 1950–1968 Prof. at Trondheim and 1968–1977 the newly created chair for Mathematical Methods in the Sciences and Numerics at Heidelberg; contributions to the Numerics of differential equations and numerical integration ("Romberg method").

The basis of approximating $\lim_{h\to 0} a(h)$ by extrapolation is again an asymptotic expansion $a(h)$ with respect to powers of h.

Theorem 6.2 (Euler-Maclaurin summation formula): *For $f \in C^{2m+2}[a,b]$ there holds the following so-called "Euler[7]-Maclaurin[8] summation formula":*

$$a(h) = \int_a^b f(x)\,dx + \sum_{k=1}^{m} h^{2k} \frac{B_{2k}}{(2k)!} \Big\{ f^{(2k-1)}(b) - f^{(2k-1)}(a) \Big\} + h^{2m+2} R_{m+1}, \qquad (6.2.17)$$

with the with the so-called Bernoulli numbers B_{2k} and a remainder term of the form

$$h^{2m+2} R_{m+1} := h^{2m+2} \frac{b-a}{(2m+2)!} B_{2m+2} f^{(2m+2)}(\zeta)\,, \quad \zeta \in [a,b].$$

Proof: For the very techical proof see the relevant literature, e. g., Stoer [19]. Q.E.D.

The Bernoulli[9] numbers are determined, e. g., as the coefficients in the power series

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k, \qquad (6.2.18)$$

and satisfy the recursion formula

$$B_k = -\sum_{j=0}^{k-1} \frac{k!}{j!(k-j+1)!} B_j\,, \quad k = 1, \dots, \qquad (6.2.19)$$

$$B_0 = 1\,, \quad B_1 = -\tfrac{1}{2}\,, \quad B_2 = \tfrac{1}{6}\,, \quad B_4 = -\tfrac{1}{30}\,, \quad B_6 = \tfrac{1}{42}\,, \quad B_8 = -\tfrac{1}{30}\,, \dots .$$

For odd indices there holds $B_{2j+1} = 0$, and for $k \to \infty$ the Bernoulli numbers grow very fast like

$$B_{2k} \approx \frac{(2k)!}{(2\pi)^{2k}}\,.$$

The importance of the expansion (6.2.17) consists of its existence and of the information that this expansion is with respect to powers of h^2. This suggests the use of

[7]Leonhard Euler (1707–1783), born at Basel: Universal Mathematician and Physicist; most important and productive Mathematician of his time; worked at Berlin and St. Petersburg; contributions to all mathematival areas of his time.

[8]Colin Maclaurin (1698–1746): Scottish Mathematician; Prof. at the universities of Aberdeen (1717) and Edinburgh (1725); contributions to the at that time "novel" differential calculus of Newton (first systematic presentation of the corresponding "calculus" and development of the integral formula named after him (1742)), to klassical Mechanics, Geometry and Algebra.

[9]Bernoulli: Swiss family of Mathematicians; Jakob Bernoulli (1655–1705) worked at Basel; already used complete induction; discovered the so-called "Bernoulli numbers" and co-founder of probability theory; his jounger brother Johann Bernoulli (1667–1748) worked finally also at Basel and after, the death of his older brother, was considered the leading Mathematician of his time; contributed to real series, differential equations; his son Daniel Bernoulli (1700–1782) continued this work; worked at St. Petersburg and Basel; important contributions to Hydromechanics and Gasdynamics.

extrapolation with even polynomials, i. e., in h^2, as described above. As consequence of the general Theorem 6.1, we obtain the following result.

Theorem 6.3 (Romberg Integration): *Let $f \in C^{2m+2}[a,b]$. The extrapolated values $a_{m,m}$ for the step sizes $h_k = h/2^k$, $k = 0,\dots,m$, converge to $a_0 = \lim_{h\to 0} a(h)$ like*

$$a(0) - a_{m,m} = O(h^{2m+2}). \tag{6.2.20}$$

The Romberg method together with the above stopping criterion is the most efficient integration method for integrals of "very smooth" functions.

6.3 Extrapolation in the numerical solution of ODE

For applying Richardson extrapolation to the numerical solution of initial value problems (IVP) of ordinary differential equations (ODE), we need to show that the (global) discretization error $e_n := u(t_n) - y_n$ admits an asymptotic expansion with respect to the step size $h > 0$. First, we consider the approximation of an general IVP (d-dimensional system of ODE) written in compact form,

$$u'(t) = f(t, u(t)), \quad t \in I = [t_0, t_0 + T], \quad u(t_0) = u_0, \tag{6.3.21}$$

by an (explicit or implicit) one-step scheme of the form

$$y_{n+1} = y_n + h_n F(h_n, t_n; y_n, y_{n+1}), \quad n \geq 0, \quad y^0 = u^0, \tag{6.3.22}$$

with a Lipschitz continuous "process function" $F(t, h; \cdot, \cdot)$ and step sizes $h_0, h_1, \dots h_{N-1}$, $\sum_{n=0}^{N-1} = T$, $h := \max_{0\leq n\leq N-1} h_n$. This scheme is assumed to have the (local) "consistancy order" $m \geq 1$, i. e., the corresponding "trumcation error" $\tau_n^h(u)$ defined by

$$\tau_n^h(u) := h_n^{-1}(u_{n+1} - u_n) - h_h F(t_n, h_n; u_n, u_{n+1}), \quad u_n := u(t_n),$$

satisfies

$$\max_{1\leq n\leq N} \|\tau_n^h(u)\| \leq c(u) h^m. \tag{6.3.23}$$

Then, due to the L-continuity of $F(t, h; \cdot, \cdot)$, we have stability and also the (global) "convergence order" $m \geq 1$ (L_F the L-constant of the process function $F(t, h; \cdot, \cdot)$),

$$\|y_n - u(t_n)\| \leq e^{L_F(t_n - t_0)} \sum_{\nu=1}^{n} h_\nu \|\tau_\nu^h(u)\| = \mathcal{O}(h^m), \quad 0 \leq n \leq N. \tag{6.3.24}$$

Theorem 6.4 (General asymptotic expansion): *Let the function $f(t,x)$ be $(M{+}1)$-times continuously differentiable on $I \times \mathbb{R}^d$. Then, for the approximate solution y^n obtained by the scheme (6.3.22) with uniform step sizes $h_n = h$, there holds the following asymptotic expansion:*

$$y_n = u(t_n) + h^m e_m(t_n) + \ldots + h^M e_M(t_n) + h^{M+1} E_{M+1}(t_n; h)\,, \qquad (6.3.25)$$

where the functions $e_i(t)$ are independent of h and the remainder $E_{M+1}(t_n; h)$ is bounded.

Proof: We sketch the very technical proof only for the simplest (explicit) Euler scheme with $F(t, h; x, y); = f(t, x)$,

$$y_{n+1} = y_n + hf(t_n, y_n), \quad n \geq 0, \quad y_0 = u_0, \qquad (6.3.26)$$

and a scalar ODE, $d = 1$, and the basic case $m = 1$, $M = 2$. The underlying idea of proof then carries over also to more general situations. For the solution $u \in C^3(I)$ there holds by Taylor expansion:

$$u(t_{n+1}) = u(t_n) + h\, f(t_n, u(t_n)) + \tfrac{1}{2}h^2 u''(t_n) + \tfrac{1}{6}h^3\, u'''(\xi_n),$$

with an intermediate value $\xi_n \in (t_n, t_{n+1})$. For the error $e_n := y_n - u(t_n)$ it follows that

$$\begin{aligned} e_{n+1} &= e_n + h\{f(t_n, y_n) - f(t_n, u(t_n))\} - \tfrac{1}{2}h^2 u''(t_n) - \tfrac{1}{6}h^3 u'''(\xi_n) \\ &= e_n + hf'_x(t_n, u(t_n))\, e_n + \tfrac{1}{2}hf''_{xx}(t_n, \eta_n)\, e_n^2 \\ &\quad - \tfrac{1}{2}h^2 u''(t_n) - \tfrac{1}{6}h^3 u'''(\xi_n), \end{aligned}$$

with another intermediate value $\eta_n \in (u(t_n), y_n)$. Then, the values $\bar{e}_n := \frac{1}{h}e_n$ obviously satisfy the difference equation

$$\bar{e}_{n+1} = \bar{e}_n + h\{f'_x(t_n, u(t_n))\bar{e}_n - \tfrac{1}{2}u''(t_n)\} + h^2 r_n \qquad (6.3.27)$$

with
$$r_n := \tfrac{1}{2}f''_{xx}(t_n, \eta_n)\, \bar{e}_n^2 - \tfrac{1}{6}u'''(\xi_n)\,.$$

The assumed 1st-order convergence of the Euler scheme implies that

$$|e_n| \leq K_1 h,$$

with some constant $K_1 > 0$, and, consequently,

$$|r_n| \leq \tfrac{1}{2} \max_{(t,x)\in I\times\mathbb{R}} |f''_{xx}(t, x)|\, K_1^2 + \tfrac{1}{6} \max_{t\in I} |u'''(t)| =: K_2.$$

The relation (6.3.27) can be interpreted as the applicatio of a perturbed Euler scheme to the linear IVP

$$e'(t) = f_x(t, u(t))e(t) - \tfrac{1}{2}u''(t)\,, \quad t \in I, \quad e(t_0) = 0\,, \qquad (6.3.28)$$

where in each step an additional term $h^2 r_n$ is added. The proof (based on a discrete Gronwall[10] inequality) of the error estimate (6.3.24) then implies the perturbed estimate

[10]Thomas Hakon Gronwall (orig. Hakon Grönwall) (1877–1932): Swedish/US-American mathematician and civil engineer; studied in Sweden (PhD at Uppsala 1898); emigrated to the United States in 1904; worked as engineer for several compsanies; taught Mathematics at Princeton and later Physica at

$$|\bar{e}_n - e(t_n)| \le e^{L(t_n-t_0)} \Big\{ \sum_{\nu=1}^{n} h|\bar{\tau}_\nu^h(e)| + \sum_{\nu=1}^{n} h^2|r_\nu| \Big\} \le K_3 h, \qquad (6.3.29)$$

with the truncation error $\bar{\tau}_\nu^h(e)$ corresponding to the linear IVP (6.3.28). It follows that

$$e_n = h\, e(t_n) + h^2 E_2(t_n; h)\,, \quad |E_2(t_n; h)| \le K_3,$$

what was to be shown. Q.E.D.

The result of Theorem 6.4 ensures that extrapolation to the limit can be applyed to all common single-step schemes for solving IVP of ODE. Usually here extrapolation is not used globally on the whole interval $[t_0, t_n]$ but rather locally in each time step $t_n \to t_{n+1}$ in order to increase the consistency order of the chosen base scheme. The corresponding higher global convergence order then follows again by the standard argument using a discrete Gronwall inequality.

Possible low-order candidates are the 1st-order explicit Euler scheme considered in the proof of Theorem 6.4,

$$y_{n+1} = y_n + hf(t_n, y_n),$$

especially in the "non-stiff" case (i. e., only slowly oscillating solution components), and in the "stiff" case (i. e., slowly and very rapidly oscillating solution components) the likewise 1st-order implicit Euler scheme,

$$y_{n+1} = y_n + hf(t_n, y_{n+1}),$$

and the 2nd-order Trapezoidal rule,

$$y_{n+1} = y_n + \tfrac{1}{2}h\big\{f(t_n, y_n) + f(t_{n+1}, y_{n+1})\big\},$$

both A-stable methods. In choosing a "base method" for the extrapolation process, we have to observe the following criteria:

- The scheme should be as simple as possible, i. e., only 1 function evaluation per time step since the higher accuracy is to be obtained by the extrapolation process.
- For the sake of efficiency the scheme should allow for an error expansion with respect to even powers of h. This requores to use symmetrically placed function evaluation.
- The scheme should be "explicit" in the non-stiff case.

These conditions rule out the above simple schemes as base scheme for extrapolation in the non-stiff case. Only in the stiff case the A-stable trapezoidal rule may be a suitable candidate (see the remark below).

Columbia University; contributions to classical Analysis and Differential Equations, but also to Physical Chemsitry and Atomic Phsics; in 1919 he proved the well-known differential inequality named after him.

6.3.1 The non-stiff case: Gragg's extrapolation method

The remedy of the delemma indicated above is to widen the view and to look also for two-step schemes. There, we have the explicit, 2nd-order "midpoint rule",

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n), \tag{6.3.30}$$

which meets all of the above requirement: explicit, only 1 function evaluation per time step, symmetric placement of the evaluation point. In deed, it appears that this is the only available candidate for the choice of a "base scheme in the non-stiff case. For this scheme, we have the following result of W. B. Gragg[11] (1963):

Theorem 6.5 (Theorem of Gragg): *Let the function $f(t,x)$ be $(2m+2)$-times continuously differentiable on $I \times \mathbb{R}^d$. Then, the approximations y^n obtained by the midpoint rule with one explicit Euler step as starting procedure and uniform step size h admits the following asymtotic error expansion:*

$$y_n = u(t_n) + \sum_{k=1}^{m} h^{2k}\{a_k(t_n) + (-1)^n b_k(t_n)\} + h^{2m+2} E_{2m+2}(t_n; h)\,, \tag{6.3.31}$$

where the functions $a_k(t)$, $b_k(t)$ are independent of h and the remainder $E_{2m+2}(t;h)$ is bounded.

Proof: See W. B. Gragg: "On extrapolation algorithms for ordinary initial value problems", J. SIAM Numer. Anal., Ser. B 2, 384–403 (1965). Q.E.D.

In view of the oscillatory terms $(-1)^n b_k(t_n)$ in (6.3.31) this is not quite an asymptotic expansion of the desired form (6.3.25). The use of the explicit Euler step as starting procedure is essential for the expansion (6.3.31) to hold. Using, e. g., instead a higher-order one-step method would change the expansion into one in which all powers of h occur. The oscillatory character of the first term in the expansion (6.3.31) is related to the well-known defect of the midpoint rule to have a trivial stability region. The critical oscillatory term $(-1)^n b_1(t_n)$ in the leading error term,

$$y_n - u(t_n) = h^2 \left[a_1(t_n) + (-1)^n b_1(t_n)\right] + \mathcal{O}(h^4),$$

can be removed by a simple trick devised in Graqq's thesis (1964). After having computed the basic sequence of midpoint values y_n, for uniform step size h, one forms the averages

$$\tilde{y}_n = \tfrac{1}{4}\, y_{n+1} + \tfrac{1}{2}\, y_n + \tfrac{1}{4}\, y_{n-1}\,, \tag{6.3.32}$$

[11] William B. Gragg (1936–2016): US-Amerikanican applied Mathematician; Prof. at the Naval Postgraduate School (Monterey, California, USA); student of P. Henrici; important contributions to Numerical Linear Algebra and Numerics of ODE; the extrapolation-based method named after him was subject of his PhD thesis 1964 at the Univ. of California (UCLA, USA).

and uses those in the extrapolation process. For these averaged values then there holds:

$$\begin{aligned}\tilde{y}_n = \tfrac{1}{2}\Big\{ & u(t_n) + \tfrac{1}{2}\{u(t_{n+1}) + u(t_{n-1})\} \\ & + \sum_{k=1}^{m} h^{2k}\left[a_k(t_n) + \tfrac{1}{2}\{a_k(t_{n+1}) + a_k(t_{n-1})\}\right. \\ & + (-1)^n \left\{ b_k(t_n) - \tfrac{1}{2}\{b_k(t_{n+1}) + b_k(t_{n-1})\} \right] \Big\} + O(h^{2m+2})\,.\end{aligned}$$

Expanding here $u(t_{n\pm1})$ and $a_k(t_{n\pm1})$, $b_k(t_{n\pm1})$ into Taylor series in h yields an asymptotic error expansion of the form:

$$\begin{aligned}\tilde{y}_n = u(t_n) & + h^2\,[\,a_1(t_n) + \tfrac{1}{2}\,u^{(2)}(t_n)\,] \\ & + \sum_{k=2}^{m} h^{2k}\,[\,\tilde{a}_k(t_n) + (-1)^n \tilde{b}_k(t_n)\,] + O(h^{2m+2})\,,\end{aligned} \tag{6.3.33}$$

the leading term of which does not contain an oscillatory component anymore. The extrapolation method based on the averaged midpoint rule (6.3.32) is the most often used method of this kind for non-stiff IVP. In view of the oscillatorry higher-order terms in the expansions (6.3.31) and (6.3.33), one has to observe to use step-size sequences

$$h_i = \frac{h}{n_i}\,, \quad 1 \le n_0 < n_1 < n_2 < \dots$$

either only with even or only with odd n_i, such that $(-1)^n$ appears with the same sign. A popular sequence has been proposed by R. Bulirsch[12]:

$$\{2, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, \dots\}\,,$$

where, one usually does not use more than $6-8$ extrapolation steps. The likewise suitable “Romberg sequence” $\{2, 4, 8, 16, \dots\}$ is less efficient because of the rapidly growing work for the computation of the extrapolation table. The resulting extrapolation method named after Gragg-Stoer[13]-Bulirsch proceeds as follows:

1. Choose a base step length h for calculting the approximations

$$y_n \approx u(t_n) \quad (n = 0, 1, 2, \dots).$$

[12]Roland Bulirsch (1932–): Deutscher Numeriker; 1967–1969 Assoc. Prof. an der University of California in San Diego, USA, 1969 Prof. an der Univ. Köln und ab 1973 Prof. an der TU München; Beiträge vor allem zur Numerik von gew. Differentialgleichungen und Optimierungsaufgaben; besonders bekannt durch die von ihm entwickelte "‘Mehrzielmethode (Mehrfachschießverfahren)"’ und sein Lehrbuch "‘Numerische Mathematik"’ zus. mit J. Stoer.

[13]Josef Stoer (1934–): Deutscher Numeriker; Professor an der Univ. Würzburg; Beiträge zur Approximationstheorie, Numerik gew. Differentialgleichungen und Optimierungsverfahren; bekannt durch sein Lehrbuch "‘Numerische Mathematik zus. mit R. Bulirsch"’ basierend auf Vorlesungen seines Doktorvaters Friedrich L. Bauer (Univ. Mainz).

2. Let y_n be calculated. Choose integers $n_0 < n_1 < \ldots < n_m$ and calculate the approximations

$$\eta(t_n + \nu h_i; h_i)\,, \quad h_i = h/n_i\,, \quad \nu = 1, \ldots, n_i + 1\,,$$

using the midpoint rulel started by one explicit Euler scheme,

$$\begin{aligned} \eta(t_n + h_i; h_i) &= y_n + h_i\, f(t_n, y_n) \\ \eta(t_n + (\nu+1)h_i; h_i) &= \eta(t_n + (\nu-1)h_i; h_i) + 2h_i\, f(t_n + \nu h_i, \eta(t_n + \nu h_i; h_i)), \end{aligned}$$

and set

$$a(h_i) := \tilde{\eta}(t_{n+1}; h_i) = \tfrac{1}{4}\big\{\eta(t_{n+1} - h_i; h_i) + 2\eta(t_{n+1}; h_i) + \eta(t_{n+1} + h_i; h_i)\big\}.$$

3. Compute the diagonal entries a_{ii} of the extrapolation table using the recursion formula (6.2.12). Set $y^{n+1} := a_{mm}$ and start again at (2).

Remark 6.2: The Gragg extrapolation method can be viewed as an explicit one-step scheme with base step size h for solving the IVP (6.3.21). Due to the averaging applies to the originally numerically unstable midpoint rule the resulting scheme has a non-trivial stability region similar to that of the classical Taylor or Runge-Kutta methods. In view of Theorem 6.1 its local accuracy (consistency) for "exact" initial value y_n is $O(h^{2m+2})$. Using the index sequence $\{2,4,6,8,12,16\}$ $(m=5)$ yields a scheme of order $2m+2 = 12$.

Remark 6.3: In extrapolation to the limit, one may also use rational functions

$$T_{ik}(h) = \frac{P_{ik}(h)}{Q_{ik}(h)}\,,$$

for interpolating the values $a(h_i), \ldots, a(h_{i+k})$, which often yield better results than simple polynomials. For the values $T_{ik} = T_{ik}(0)$ of the corresponding extrapolation table there exist again recursion formula similar to those stated above for polynomial extrapolation. In the context of Richardson extrapolation this has been investigated in early work by R. Bulirsch and J. Stoer (1964).

6.3.2 A numerical example

For the non-stiff model IVP

$$u'(t) = -200t\, u(t)^2\,, \quad t \geq -3, \qquad u(-3) = 1/901\,,$$

with the (unique) solution $u(t) = (1+100t^2)^{-1}$ we have approximated the value $u(0) = 1$ by the Gragg extrapolation method.

For $i = 0, \dots, m$):

$$\begin{aligned}
\eta(t_n + h_i; h_i) &= y_n + h_i f(t_n, y_n) \\
\eta(t_n + (\nu+1)h_i; h_i) &= \eta(t_n + (\nu-1)h_i; h_i) + 2h_i f(t_n + \nu h_i, \eta(t_n + \nu h_i, h_i)) \\
&\quad (\nu = 1, \dots, n_i + 1) \\
a(h_i) &= \tfrac{1}{4}\big\{\eta(t_{n+1} - h_i; h_i) + 2\eta(t_{n+1}; h_i) + \eta(t_{n+1} + h_i; h_i)\big\}
\end{aligned}$$

and then

$$T_{i0} := a(h_i)\,, \quad T_{ik} = T_{i,k-1} + \frac{T_{i,k-1} - T_{i-1,k-1}}{(h_i/h_{i+k})^2 - 1}\,,$$
$$(i = 0, \dots, m+1;\ k = 1, \dots, m)$$
$$y_{n+1} = T_{mm}\,, \quad U_{mm} = 2T_{m+1,m} - T_{mm}.$$

The step size control is organized according to the criterion

$$h^{-1}\,|T_{mm} - U_{mm}| \sim \varepsilon = \text{eps}\,|y_n|/h\,.$$

Using the "Bulirsch sequence" $\{h/2, h/4, h/6, h/8\}$ $(m = 4)$ with $h = 0.1$, we obtained (16-digits arithmetic) the following results: sich mit 17-stelliger Rechnung die folgenden Resultate:

Table 6.1: Results obtained by Gragg's method for the model IVP.

Order	eps	$h_{\min}$	$h_{\max}$	Error	Evaluations
$2m + 2 = 10$	10^{-13}	$6 \cdot 10^{-3}$	0.1	$2 \cdot 10^{-12}$	~ 7.800
Computation with constant mean step length					
		$2.5 \cdot 10^{-2}$		$6 \cdot 10^{-12}$	~ 4.400

The adaptive step-size selection requires additional work but it guarantees to match the error tolerance eps $\sim 10^{-12}$ without requiring a priori knowledge of the "optimal" uniform step size $h = 2.5 \cdot 10^{-2}$.

6.3.3 The stiff case

The applixation of extrapolation in the context of "stiff" IVP requires the use of a simple A-stable base scheme. Candidates are here mainly the only 1st-order *implicit* Euler scheme oder the 2nd-order trapezoidal rule stated above. The latter again admits for an asymptotic error expansion in h^2. However, due to its lacking *strong* A-stability the trapezoidal rule is sensitiv with respect to data perturbations, so that in practice the more robust implicit Euler scheme is preferred. It can be shown that the extrapolated Euler scheme still has good stability properties.

To this end, we recall the basics of "numerical stability theory" (see the Lecture Notes "Numrik 1 (Numerik gewöhnlicher Differentialgleichungen)"). A one-step difference scheme is called "numerically stable" for some step length h if it produces bounded approximations as long as the exact solution stays bounded. This concept can be formalized by checking the behavior pif the scheme when applied to the scalar model equation

$$u'(t) = \lambda u(t), \quad t \ge t_0, \tag{6.3.34}$$

for any $\lambda \in \mathbb{C}$. Here, the parameter $\lambda \in \mathbb{C}$ stands for the (possibly complex) eigenvalues of the Jacobian matrix $f'_x(t_*, u(t_*)$ of the right-hand side function $f(t,x)$ in the IVP along the solution $u(t)$ frozen at any time $t_* \ge t_0$. The concept of (linearized) "numerical stability" is based on the idea that a scheme, which is stable for the model problem is also stable in the general case. This conjecture can be theoretically justified to some extent if the Jacobian $f'_x(t_*, u(t_*)$ is diagonalizable (i. e., possesses a basis of eigenvectors). Counterexamples show that this may not be true if $f'_x(t_*, u(t_*)$ is not diagonalizable. The exact solution of the model equation is $u(t) = e^{\lambda t}$, which stays bounden for $\operatorname{Re}\lambda \le 0$ and growth exponentially otherwise. Therefore, the investigation of numerical stability can be restricted to parameter $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \le 0$. We are interested in the values $z = h\lambda$ for which the scheme considred is stable. This gives rise to the definition of the "stability region"

$$SG := \{z = h\lambda \in \mathbb{C} : \text{ the scheme produced bounded solutions}\},$$

and of the "stability interval"

$$SI := \{z = h\lambda \in \mathbb{R} : \text{ the scheme produced bounded solutions}\},$$

the latter being relevant particularly if the eigenvalues of $f'_x(t_*, u(t_*)$ are real, e. g., if $f'_x(t_*, u(t_*)$ is symmetric. A difference scheme is called "A-stable" if its stability region contains the entire left complex half-plane, i. e., it is stable for all step sizes h and all values λ for which the exact solution stays bounded.

Applied to the model equation (6.3.34) all practical one-step schemes take the form

$$y_n = R(h\lambda) y_{n-1}, \quad n \ge 1, \tag{6.3.35}$$

with some rational function

$$R(z) = \frac{P(z)}{Q(z)}, \quad z \in \mathbb{C}.$$

For the two schemes of interest, we have:

i) implicit Euler scheme:

$$R(z) = \frac{1}{1-z},$$

ii) trapezoidal rule:

$$R(z) = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}.$$

With this notation, we have

$$|y_n| \le |\omega(h\lambda)|\, |y_{n-1}| \le |\omega(h\lambda)|^n\, |y_0|,$$

with the "amplification factor" $\omega(z) := R(z)$. From this, we conclude that

$$SG = \{z \in \mathbb{C} : \ |\omega(z)| \le 1\}, \quad SI = \{z \in \mathbb{R} : \ |\omega(z)| \le 1\}.$$

We see that both schemes, implicit Euler scheme and trapezoidal rule, are A-stable. From the relation

$$\Big|\frac{1}{1-z}\Big| \le 1 \quad \Leftrightarrow \quad |1 - z| \ge 1,$$

we see that the stability region of the implicit Euler scheme is the complement of the (open) unit circle $K_1(1)$, i. e., this method is "over-stable" since it is stable even for values λ with $\operatorname{Re}\lambda \ge 1$. The trapezoidal rule turns out to be perfectly A-stable since it is stable exactly for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \le 1$.

The extrapolated implicit Euler scheme can be viewed as an implicit one-step scheme with the base step size h. It can be shown (exercise) that this scheme is also A-stable and can therefore be used for stiff IVP.

6.4 Extrapolation in the FE discretization of elliptic PDE

For illustrating the argument used in the proofs of asymptotic error expansions for finite element methods, we consider the usual Poisson model problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega. \tag{6.4.36}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded domain with a sufficiently regular boundary $\partial\Omega$ and the right-hand side f is assumed to be smooth. First, suppose that Ω is a convex polygonal domain. Then, the weak solution $u \in H_0^1(\Omega)$ of problem (6.4.36) satisfies $u \in H^2(\Omega)$ and is smooth in any subdomain $\Omega' \subset\subset \Omega$ having positive distance to the corner points of $\partial\Omega$. Further, there holds the a priori estimate

$$\|u\|_{H^2} \le c\|f\|. \tag{6.4.37}$$

For some mesh size parameter $h \in \mathbb{R}_+$ let $\{\mathbb{T}_h\}_h$ be a strongly regular family of triangulations of $\overline{\Omega}$, i. e., each triangle $T \in \mathbb{T}_h$ has a diameter $h_T \le c_1 h$ and contains a circle of radius $\rho_T \ge c_2 h$ (see Ciarlet [31]). By $\partial\mathbb{T}_h$ we denote the union of all edges of the triangles in $\mathbb{T}_h$ and write $C_0(\partial\mathbb{T}_h)$ for the space of all continuous functions on $\partial\mathbb{T}_h$

which vanish along $\partial\Omega$. Let

$$V_h := \{v_h \in H_0^1(\Omega) : \ v_h|_T \in P_1(T),\ T \in \mathbb{T}_h\}$$

be the associated spaces of (cellwise) linear finite elements. Then, the Ritz projection $R_h u \in V_h$ of u is determined through the equation

$$(\nabla R_h u, \nabla\varphi_h) = (\nabla u, \nabla\varphi_h) = (f, \varphi_h) \quad \forall\varphi_h \in V_h. \tag{6.4.38}$$

This may equivalently be written as a linear algebraic system for the coefficients $x_i = R_h u(P_i)$ in the representation of $R_h u$ in terms of the usual nodal basis of V_h. For the error in the discretization (6.4.38) there hold the optimal-order L^2-error estimates

$$\|u - R_h u\| + h\|\nabla(u - R_h u)\| \le ch^2\|u\|_{H^2} \tag{6.4.39}$$

(see Ciarlet [31]) and the "interior" L^∞-error estimate

$$\|u - R_h u\|_{L^\infty(\Omega'')} \le ch^2 L(h)\{\|u\|_{H^{2,\infty}(\Omega')} + \|u\|_{H^2(\Omega)}\}, \tag{6.4.40}$$

where $L(h) := |\ln(h)| + 1$ and $\Omega'' \subset \Omega' \subset \Omega$ are fixed interior subdomains, which have increasing positive distance to the corner points of the polygonal domain Ω. If $u \in H^{2,\infty}(\Omega)$ (an unrealistic assumption on a general polygonal domain), then these estimates hold true with $\Omega'' = \Omega' = \Omega$. A proof of (6.4.40) will be given below.

6.4.1 General concept for deriving error expansions

For deriving error expansions in the context of finite element methods, the argument essentially proceeds as follows. Suppose, for simplicity, that the family of triangulations $\{\mathbb{T}_h\}_h$ is "equi-three-directional", i. e., all sides of triangles are parallel to three fixed directional unit vectors t_i, $i = 1, 2, 3$ (see Fig. 6.2). Further, suppose that $u \in C^4(\overline{\Omega})$ (This assumption can substantially be weakened.).

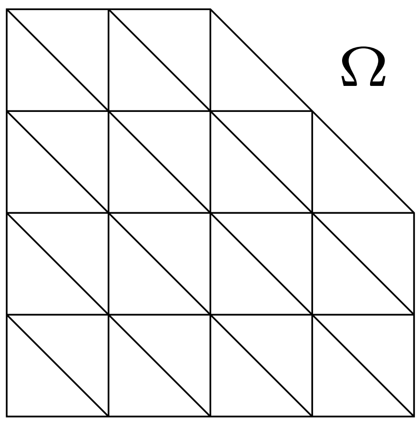

Figure 6.2: A three-directional mesh

In virtue of the Galerkin-orthogonality relation

$$(\nabla(u - R_h u), \nabla\varphi_h) = 0, \quad \varphi_h \in V_h,$$

there holds

$$(\nabla(R_h u - I_h u), \nabla\varphi_h) = (\nabla(u - I_h u), \nabla\varphi_h), \quad \varphi_h \in V_h, \tag{6.4.41}$$

where I_h denotes the usual nodal interpolation operator onto V_h. Using the uniform structure of the triangulations the term on the right in (6.4.41) may be expanded like

$$(\nabla(u - I_h u), \nabla\varphi_h) = h^2(\nabla e, \nabla\varphi_h) + h^4\rho_h(u;\varphi_h), \quad \varphi_h \in V_h, \tag{6.4.42}$$

with a certain function $e \in H_0^1(\Omega)$ independent of h. Below, we will show that the remainder term $\rho_h(u;\varphi_h)$ can be written in the form

$$\rho_h(u;\varphi_h) = \sum_{T\in\mathbb{T}_h} \int_{\partial T} r_h(u)\partial_n\varphi_h\, ds, \tag{6.4.43}$$

with a function $r_h(u) \in C_0(\partial\mathbb{T}_h)$ satisfying $\|r_h(u)\|_{L^\infty} \le c\|u\|_{H^{4,\infty}}$. Setting

$$v_h := R_h u - I_h u - h^2 R_h e,$$

we are led to

$$(\nabla v_h, \nabla\varphi_h) = h^4 \sum_{T\in\mathbb{T}_h} \int_{\partial T} r_h(u)\partial_n\varphi_h\, ds \quad \varphi_h \in V_h. \tag{6.4.44}$$

We want to interpret this relatiobn in the sense that $v_h = h^4 R_h r_h(u)$. This requires to extend the domain of definition of the Ritz projection R_h to the space $C_0(\partial\mathbb{T}_h)$. Integration by parts in (6.4.38) yields

$$(\nabla R_h u, \nabla\varphi_h) = \sum_{T\in\mathbb{T}_h} \int_{\partial T} u\, \partial_n\varphi_h\, ds, \quad \varphi_h \in V_h.$$

Clearly, by this identity R_h is well defined on functions in the space $H_0^1(\Omega) \cup C_0(\partial\mathbb{T}_h)$. In this sense, we then have that

$$R_h u - I_h u = h^2 R_h e + h^4 R_h r_h(u). \tag{6.4.45}$$

Now, in order to derive an expansion for $R_h u - I_h u$ from (6.4.45), we would like to use an a priori bound for the extended Ritz projection of the form

$$\|R_h v\|_{L^\infty} \le c(h)\|v\|_{C_0(\partial\mathbb{T}_h)}, \quad v \in C_0(\partial\mathbb{T}_h). \tag{6.4.46}$$

In fact, such an estimate holds true with $c(h) = cL(h)$. This will be derived using the L^1-error estimates for discrete Green functions given in Chapter 3 of the Lecture

Notes “Special Topics in Numerics I”. For linear elements the logarithmic factor in the stability constant is unavoidable in general, what is shown by numerical experiments and theoretical analysis.

In view of (6.4.46) and (6.4.40) applied for $r_h(u)$, we obtain by (6.4.44) the expansion

$$R_h u(P) = u(P) + h^2 e(P) + \mathcal{O}(h^4 L(h)) \tag{6.4.47}$$

at nodal points P. This result requires the solution u to have C^4-regularity up to the corners of $\partial\Omega$, which is an unrealistic assumption in general. However, on the basis of (6.4.47) it can be shown by an approximation argument (see Lin & Zhu [91]) that

$$R_h u(P) = u(P) + h^2 e(P) + o(h^2), \tag{6.4.48}$$

remains valid at interior nodal point $P \in \Omega' \subset\subset \Omega$ if the solution has only the “minimal” regularity $u \in H^2(\Omega) \cap C^3(\Omega)$.

6.4.2 Expansion of the consistency error

The first step in deriving asymptotic error expansions of the form (6.4.47) is the derivation of expansions (6.4.42) of the consistancy error. For an arbitrary but fixed triangle $T \in \mathbb{T}_h$ the following notation is used:

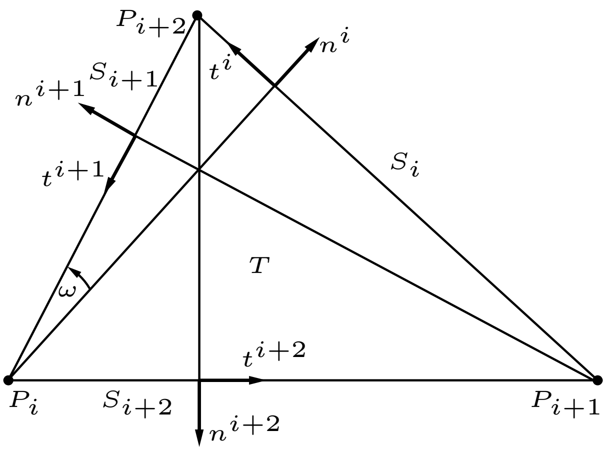

Figure 6.3: Triangle scheme

P_i : vertices of T
S_i : side of T opposite to P_i,
$h_i = \lambda_i h$: length of S_i,
H_i : height of P_i over S_i,
$A = \alpha h^2$: area of T,
n^i, t^i : outer normal and tangent unit vectors along S_i,
$D_i = t_i \cdot \nabla$: directional derivative,
(The indices $i.i+1, i+2$ have to be understood “mod 3”.),
N_i : nodal basis function corresponding to P_i, $N_i(P_j) = \delta_{ij}$.

Lemma 6.1: *On the triangle T there holds:*

$$\sum_{i=1}^{3} \nabla N_i \equiv 0, \qquad \nabla N_i = -\frac{h_i}{2A} n^i, \quad i = 1,2,3, \tag{6.4.49}$$

$$t^{i+1} \cdot n^i = -\frac{2A}{h_i h_{i+1}}, \qquad t^i \cdot n^{i+1} = \frac{2A}{h_i h_{i+1}}, \quad i = 1,2,3, \tag{6.4.50}$$

where the index $i+1$ is used mod(3).

Proof: i) By construction of the nodal basis functions there holds $\sum_{i=1}^3 N_i \equiv 1$, which implies the first identity in (6.4.49).

ii) Since $N_i(P_j) = \delta_{ij}$, we have $D_i N_i \equiv 0$ and therefore $\nabla N_i \equiv -H_i^{-1} n^i$. Hence, observing $A = \frac{1}{2} H_i h_i$, we obtain the second identity in (6.4.49).

iii) In view of the relations $H_i = h_{i+1}\cos(\omega)$ and $\cos(\omega) = -t^{i+1}\cdot n^i$ there holds $A = \frac{1}{2} h_i H_i = -\frac{1}{2} h_i h_{i+1} t^{i+1}\cdot n^i$. which implies the first identity in (6.4.50). The other follows by analogous arguments. Q.E.D.

We consider the integrals

$$I_\Omega := (\nabla(u - I_h u), \nabla\varphi_h) = \sum_{T\in\mathbb{T}_h} (\nabla(u - I_h u), \nabla\varphi_h)_T =: \sum_{T\in\mathbb{T}_h} I_T,$$

for a sufficiently smooth solution u and any $\varphi_h \in V_h$. Integration by parts and observing that $\varphi_h|_T \in P_1(T)$ gives

$$I_T = \sum_{i=1}^3 \partial_{n^i}\varphi_h \int_{S_i} (u - I_h u)\, ds =: \sum_{i=1}^3 I_T^i\,.$$

By the Euler-MacLaurin summation formula there holds (see Section 6.2.4)

$$\int_{S_i} (u - I_h u)\, ds = \sum_{k=1}^{n-1} \beta_k h_i^{2k} \int_{S_i} D_i^{2k} v\, ds + h_i^{2n} \int_{S_i} \beta_n^i(s) D_i^{2n} v\, ds,$$

where $\beta_k\,(k = 1,\dots,n-1)$ are certain constants independent of h_i (related to the "Bernoulli numbers"), and $\beta_n(\cdot) \in C_0^1(S_i)\cap C^2(S_i)$. This expansion provides the basis for deriving error expansions of higher order. Here, we concentrate on the lowest-order case $n = 2$, in which $\beta_1 = -\frac{1}{12}$ and $\beta_2^i(s) = \frac{1}{24} s^2 h_i^{-2}(1 - s h_i^{-1})^2$,

$$\int_{S_i} (u - I_h u)\, ds = \beta_1 h_i^2 \int_{S_i} D_i^2 u\, ds + h_i^4 \int_{S_i} \beta_2^i(s) D_i^4 u\, ds\,. \tag{6.4.51}$$

This implies that

$$I_T^i = \partial_{n^i}\varphi_h \beta_1 h_i^2 \int_{S_i} D_i^2 u\, ds + h_i^4 \int_{S_i} \beta_2^i D_i^4 u\, \partial_{n^i}\varphi_h\, ds\,. \tag{6.4.52}$$

Next, we note that on the side S_i, observing (6.4.49), there holds

$$\begin{aligned}
\partial_{n^i}\varphi_h = n^i\cdot\nabla\varphi_h &= \sum_{j=1}^3 \varphi_h(P_j) n^i\cdot\nabla N_j \\
&= \big\{\varphi_h(P_i) - \varphi_h(P_{i+1})\big\} n^i\cdot\nabla N_i + \big\{\varphi_h(P_{i+2}) - \varphi_h(P_{i+1})\big\} n^i\cdot\nabla N_{i+2} \\
&= -h_{i+2} D_{i+2}\varphi_h n^i\cdot\nabla N_i + h_i D_i\varphi_h n^i\cdot\nabla N_{i+2} \\
&= \frac{h_i h_{i+2}}{2A}\big\{D_{i+2}\varphi_h - n^i\cdot n^{i+2} D_i\varphi_h\big\}.
\end{aligned}$$

Using this relation in (6.4.52) leads to

$$\begin{aligned} I_T^i = \beta_1 \frac{h_i^3 h_{i+2}}{2A} \int_{S_i} D_{i+2}\varphi_h D_i^2 u\, ds - \beta_1 \frac{h_i^3 h_{i+2}}{2A} \int_{S_i} n^i \cdot n^{i+2} D_i\varphi_h D_i^2 u\, ds \\ + h_i^4 \int_{S_i} \beta_2^i D_i^4 u\, \partial_n \varphi_h\, ds\,. \end{aligned} \tag{6.4.53}$$

For converting the first line integral on the right in (6.4.51) into an area integral, we provide the following lemma.

Lemma 6.2: *For $v \in C(T)$, there holds*

$$\int_{S_i} v\, ds = \frac{h_i}{h_{i+2}} \int_{S_{i+2}} v\, ds - \frac{h_i h_{i+1}}{2A} \int_T D_{i+1} v\, dx, \quad i = 1,2,3. \tag{6.4.54}$$

Proof: Using the theorem of Gauß, we have

$$\int_T D_{i+1} v\, dx = \int_T t^{i+1} \cdot \nabla v\, dx = \int_{\partial T} t^{i+1} \cdot n\, v\, ds = \int_{S_i} t^{i+1} \cdot n^i v\, ds + \int_{S_{i+2}} t^{i+1} \cdot n^{i+2} v\, ds,$$

and hence, observing (6.4.50),

$$\int_T D_{i+1} v\, dx = -\frac{2A}{h_i h_{i+1}} \int_{S_i} v\, ds + \frac{2A}{h_{i+1} h_{i+2}} \int_{S_{i+2}} v\, ds.$$

This implies the asserted identity. Q.E.D.

We apply (6.4.54) to the first line integral in (6.4.53) obtaining

$$\int_{S_i} D_{i+2}\varphi_h D_i^2 u\, ds = \frac{h_i}{h_{i+2}} \int_{S_{i+2}} D_{i+2}\varphi_h D_i^2 u\, ds - \frac{h_i h_{i+1}}{2A} \int_T D_{i+2}\varphi_h D_{i+1} D_i^2 u\, dx\,,$$

and consequently,

$$\begin{aligned} I_T^i = \beta_1 \frac{h_i^4}{2A} \int_{S_{i+2}} D_{i+2}\varphi_h D_i^2 u\, ds - \beta_1 \frac{h_i^4 h_{i+1} h_{i+2}}{4A^2} \int_T D_{i+2}\varphi_h D_{i+1} D_i^2 u\, dx \\ + \beta_1 \frac{h_i^3 h_{i+2}}{2A} \int_{S_i} n^i \cdot n^{i+2} D_i\varphi_h D_i^2 u\, ds + h_i^4 \int_{S_i} \beta_2^i D_i^4 u\, \partial_{n^i} \varphi_h\, ds\,. \end{aligned} \tag{6.4.55}$$

For the area integral, integration by parts observing $t^{i+2} \cdot n^{i+2} = 0$ and the identities (6.4.50) yields the following identities:

$$\begin{aligned}\int_T D_{i+2}\varphi_h D_{i+1}D_i^2 u\,dx &= -\int_T \varphi_h D_{i+2}D_{i+1}D_i^2 u\,dx + \int_{S_i} t^{i+2}\cdot n^i \varphi_h D_{i+1}D_i^2 u\,ds \\ &\quad + \int_{S_{i+1}} t^{i+2}\cdot n^{i+1}\varphi_h D_{i+1}D_i^2 u\,ds \\ &= -\int_T \varphi_h D_{i+2}D_{i+1}D_i^2 u\,dx + \frac{2A}{h_i h_{i+2}}\int_{S_i}\varphi_h D_{i+1}D_i^2 u\,ds \\ &\quad - \frac{2A}{h_{i+1}h_{i+2}}\int_{S_{i+1}}\varphi_h D_{i+1}D_i^2 u\,ds\,.\end{aligned}$$

Inserting this into (6.4.55), we obtain

$$\begin{aligned} I_T^i &= \beta_1\frac{h_i^4}{2A}\int_{S_{i+2}} D_{i+2}\varphi_h D_i^2 u\,ds - \beta_1\frac{h_i^4 h_{i+1}h_{i+2}}{4A^2}\Big\{ -\int_T \varphi_h D_{i+2}D_{i+1}D_i^2 u\,dx \\ &\quad + \frac{2A}{h_i h_{i+2}}\int_{S_i}\varphi_h D_{i+1}D_i^2 u\,ds - \frac{2A}{h_{i+1}h_{i+2}}\int_{S_{i+1}}\varphi_h D_{i+1}D_i^2 u\,ds\Big\} \\ &\quad - \beta_1\frac{h_i^3 h_{i+2}}{2A}\int_{S_i} n^i\cdot n^{i+2} D_i\varphi_h D_i^2 u\,ds + h_i^4\int_{S_i}\beta_2^i D_i^4 u\,\partial_{n^i}\varphi_h\,ds \\ &= \beta_1\frac{h_i^4}{2A}\int_{S_{i+2}} D_{i+2}\varphi_h D_i^2 u\,ds + \beta_1\frac{h_i^4 h_{i+1}h_{i+2}}{4A^2}\int_T \varphi_h D_{i+2}D_{i+1}D_i^2 u\,dx \\ &\quad - \beta_1\frac{h_i^3 h_{i+1}}{2A}\int_{S_i}\varphi_h D_{i+1}D_i^2 u\,ds + \beta_1\frac{h_i^4}{2A}\int_{S_{i+1}}\varphi_h D_{i+1}D_i^2 u\,ds \\ &\quad - \beta_1\frac{h_i^3 h_{i+2}}{2A}\int_{S_i} n^i\cdot n^{i+2} D_i\varphi_h D_i^2 u\,ds + h_i^4\int_{S_i}\beta_2^i D_i^4 u\,\partial_{n^i}\varphi_h\,ds\,. \end{aligned} \tag{6.4.56}$$

This representation holds on each single triangle $T\in\mathbb{T}_h$. Now, to get the desired expansion of the consistency error, we have to perform the summation

$$(\nabla(u-I_h u),\nabla\varphi_h) = I_\Omega = \sum_{T\in\mathbb{T}_h}\sum_{i=1}^{3} I_T^i. \tag{6.4.57}$$

From now on, we assume the mesh $\mathbb{T}_h$ to be "three-directional", i. e., the sides of the triangles in $\mathbb{T}_h$ are parallel to three fixed directional unit vectors t^i, $i=1,2,3,$. With some mesh size parameter $h\in\mathbb{R}_+$ (e. g., $h:=\max_{T\in\mathbb{T}_h}\operatorname{diam}(T)$), let $h_i=\lambda_i h$ and $A=\alpha h^2$ (see Fig. 6.4).

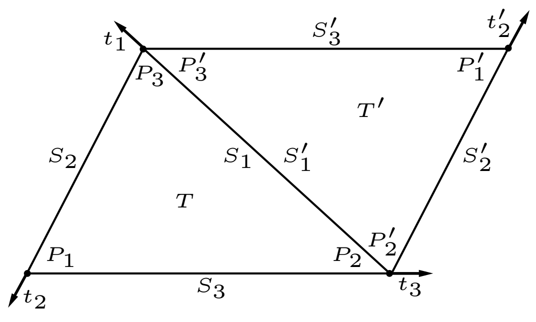

Figure 6.4: Two adjacent triangles forming a parallelogram

Summing according to (6.4.57), the followeing simplifications occur:

i) All line integrals in the first sum of the type

$$\int_{S_i} D_i\varphi_h \dots ds, \qquad \int_{S_i} \varphi_h \dots ds$$

over interior sides S_i are cancelled, since $D_i = -D_i'$. The remaining boundary integrals also vanish since $\varphi_h = 0$ along $\partial\Omega$.

ii) The area integrals combine to

$$\begin{aligned} h^2 e_h(u,\varphi_h) &:= \beta_1 \sum_{T\in\mathbb{T}_h}\sum_{i=1}^{3} \frac{h_i^4 h_{i+1}h_{i+2}}{4A^2}\int_T \varphi_h D_{i+2}D_{i+1}D_i^2 u\,dx \\ &:= h^2\beta_1\frac{\lambda_1\lambda_2\lambda_3}{4\alpha^2}\int_\Omega \varphi_h\Big(D_1D_2D_3\sum_{i=1}^{3}\lambda_i^3 D_i\Big)u\,dx\,. \end{aligned} \tag{6.4.58}$$

iii) The remainder terms add up to

$$\begin{aligned} h^4\rho_h(u,\varphi_h) &:= h^4\sum_{T\in\mathbb{T}_h}\Big(\sum_{i=1}^{3}\lambda_i^4\int_{S_i}\beta_2^i D_i^4 u\,\partial_{n^i}\varphi_h\,ds\Big) \\ &= h^4\sum_{T\in\mathbb{T}_h}\int_{\partial T} D_h^{(4)}(u)\partial_n\varphi_h\,ds\,, \end{aligned} \tag{6.4.59}$$

where the function $D_h^{(4)}(u)\in C_0(\partial\mathbb{T}_h)$ is defined on the set of sides $\partial\mathbb{T}_h$ by

$$D_h^{(4)}(u)|_{S_i} := \lambda_i^4\beta_2^i D_i^4 u,$$

and satisfies the bound

$$\|D_h^{(4)}(u)\|_{C(\partial\mathbb{T}_h)} \le c\|u\|_{C^4}. \tag{6.4.60}$$

We collect these results in the following technical lemma.

Lemma 6.3: *Let the mesh $\mathbb{T}_h$ be three-directional and $u\in H_0^1(\Omega)\cap C^4(\overline{\Omega})$, then the consistency error admits the following representation*

$$(\nabla(u-I_h),\nabla\varphi_h) = h^2 e_h(u,\varphi_h) + h^4\rho_h(u,\varphi_h), \quad \varphi_h\in V_h, \tag{6.4.61}$$

with the expansion term $e_h(u,\varphi_h)$ and the remainder term $\rho_h(u,\varphi_h)$ are as defined in (6.4.58) and (6.4.59), respectively.

In order to convert the representation (6.4.61) into an asymptotic expansion of the desired type, we introduce the solution $e\in H_0^1(\Omega)\cap H^2(\Omega)\cap C^2(\Omega)$ of the auxiliary problem

$$-\Delta e = \beta_1 \frac{\lambda_1\lambda_2\lambda_3}{4\alpha^2}\Big(D_1D_2D_3\sum_{i=1}^3 \lambda_i^3 D_i\Big)u \quad \text{in } \Omega, \quad e=0 \text{ on } \partial\Omega, \tag{6.4.62}$$

Then, we can write

$$e_h(u,\varphi_h) = (\nabla e, \nabla\varphi_h), \quad \varphi_h \in V_h.$$

On the basis of Lemma 6.3 this construction leads to the following result.

Theorem 6.6: *Let the mesh $\mathbb{T}_h$ be three-directional and $u \in H_0^1(\Omega)\cap C^4(\overline{\Omega})$, then the consistency error admits the following asymptotic expansion;*

$$(\nabla(u-I_hu),\nabla\varphi_h) = h^2(\nabla e,\nabla\varphi_h) + h^4\rho_h(u,\varphi_h), \quad \varphi_h \in V_h \tag{6.4.63}$$

with the function e defined by (6.4.62) and the remainder term $\rho_h(u;\varphi_h)$ is as defined in (6.4.59) and bounded according to (6.4.60)..

Remark 6.4: In some applications, one needs expansions of "mixed derivative" terms

$$I_\Omega^{\mu\nu} := (\partial_\mu(v-I_hv),\partial_\nu\varphi_h)_\Omega, \quad \mu,\nu\in\{1,2\},$$

for any smooth function v and $\varphi_h \in V_h$. Starting from the splitting

$$I_\Omega^{\mu\nu} = \sum_{T\in\mathbb{T}_h} I_T^{\mu\nu},$$

we again obtain by integration by parts observing $\varphi_h|_T \in P_1(T)$:

$$I_T^{\mu\nu} = \int_T \partial_\mu(v-I_hv)\partial_\nu\varphi_h\,dx = \sum_{i=1}^3\int_{S_i}(v-I_hv)\,n_\mu^i\partial_\nu\varphi_h\,ds.$$

Then, analogously as above, using Euler-MacLaurin summation formula on the line integrals, one obtains after some rearrangments and integrations by parts, that (exercise)

$$I_\Omega^{\nu\mu} = h^2 e^{\mu\nu}(v,\varphi_h) + h^4\rho_h^{\mu\nu}(v,\varphi_h), \tag{6.4.64}$$

with the expansion term

$$e^{\mu\nu}(v,\varphi_h) := \beta_1\frac{\lambda_1\lambda_2\lambda_3}{4\alpha^2}\int_\Omega \varphi_h\Big(D_1D_2D_3\sum_{i=1}^3 n_\nu^i n_\mu^i\lambda_i^3 D_i\Big)v\,dx\,,$$

and the remainder term

$$\rho_h^{\mu\nu} := \sum_{T\in\mathbb{T}_h}\sum_{i=1}^3 \lambda_i^4 n_\mu^i\int_{S_i}\beta_2^i(s)D_i^4 v\,\partial_\nu\varphi_h\,ds\,.$$

Similar expansions can be derived for the lower-order consistency terms

$$(\partial_\mu(v - I_h v), \varphi_h)_\Omega, \quad \mu \in \{1,2\}, \quad \text{and} \quad (v - I_h v, \varphi_h)_\Omega\,.$$

A further extension concerns consistency terms involving variable (smooth) coefficients such as

$$(a_{\mu\nu}\partial_\mu(v - I_h v), \partial_\nu\varphi_h)_\Omega\,.$$

Remark 6.5: Starting from the derivation of the low-order representation (6.4.63), one can carry this argument further to obtain expansions of higher order (exercise):

$$(\nabla(u - I_h u), \nabla\varphi_h)_\Omega = h^2(\nabla e^{(1)}, \nabla\varphi_h)_\Omega + h^4(\nabla e^{(2)}, \nabla\varphi_h)_\Omega + h^6\rho_h(u, \varphi_h), \qquad (6.4.65)$$

where $e^{(1)}$ and $e^{(2)}$ are again determined as solutions of certain auxiliary problems and the remainder is bounded in terms of $\|u\|_{H^{6,\infty}}$.

6.4.3 Derivation of asymptotic error expansions

On the basis of the expansion result (6.4.63) for the consistency error $(\nabla(u - I_h u), \nabla\varphi_h)$, we can now derive corresponding expansions for the discretization error $(u - R_h)(P)$ at nodal point P.

Theorem 6.7: *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain and the mesh $\mathbb{T}_h$ be three-directional (see Fig. 6.3). Further let the solution be $u \in H_0^1(\Omega) \cap C^{4+\alpha}(\overline{\Omega})$ for some $\alpha > 0$. Then, the Ritz projection admits the following asymptotic error expansion in nodal points P bounded uniformly away from the corner points of Ω:*

$$R_h u(P) = u(P) + h^2 e(P) + \mathcal{O}(h^4), \qquad (6.4.66)$$

where the expansion term $e \in H_0^1(\Omega)$ is given by the auxiliary problem (6.4.62). In the neighborhood of the corner points of $\partial\Omega$ the expansion (6.4.66) remains valid but with a remainder of the reduced order $\mathcal{O}(h^{\min(4,2+\pi/\omega)})$, where $\omega \in [\pi/3, 2\pi)$ is the maximal interior angle of $\partial\Omega$; in the exceptional case $\omega = \pi/2$ the order is $\mathcal{O}(h^4 L(h))$.

Proof: We give a simplified version of the very technical proof, which gives only a remainder term of slightly reduced order $\mathcal{O}(h^4 L(h)$, where again $L(h) := |\ln(h)| + 1$. The full proof can be found in [48].

i) We start from the expansion (6.4.63) of the consistency error:

$$(\nabla(u - I_h u), \nabla\varphi_h) = h^2(\nabla e, \nabla\varphi_h) + h^4\rho_h(u, \varphi_h), \qquad (6.4.67)$$

where $e^{(1)} \in H_0^1(\Omega)$ is determined by (6.4.62) and the remainder term has the form

$$\rho_h(u, \varphi_h) = \sum_{T \in \mathbb{T}_h} \int_{\partial T} r_h(u)\partial_n\varphi_h\, ds\,,$$

with a function $r_h(u) := D_h^{(4)}(u) \in C_0(\partial\mathbb{T}_h)$ satisfying $\|r_h(u)\|_{C(\partial\mathbb{T}_h)} \le c\|u\|_{H^{4,\infty}}$. Then, using the extended Ritz projection introduced above, we have

$$\begin{aligned}(\nabla(R_hu - I_hu), \nabla\varphi_h) &= (\nabla(u - I_hu), \nabla\varphi_h) \\ &= h^2(\nabla R_he, \nabla\varphi_h) + h^4(\nabla R_hr_h(u), \nabla\varphi_h), \quad \varphi_h \in V_h,\end{aligned}$$

and therefore

$$R_hu - I_hu - h^2R_he = h^4R_hr_h(u). \tag{6.4.68}$$

Below, we will show the following L^∞-stability estimate for the extended Ritz projection:

$$\|R_hv\|_{L^\infty} \le L(h)\|v\|_{C(\partial\mathbb{T}_h)}, \quad v \in C_0(\partial\mathbb{T}_h). \tag{6.4.69}$$

Using this for the equation (6.4.68), we obtain

$$\|R_hu - I_hu - h^2R_he\|_\infty \le ch^4L(h)\|u\|_{H^{4,\infty}}. \tag{6.4.70}$$

From this it follows that in nodal points P there holds the expansion

$$R_hu(P) = u(P) + h^2R_he(P) + \mathcal{O}(h^4L(h)). \tag{6.4.71}$$

ii) However, (6.4.71) is not yet an error expansion of the desired form as the expansion coefficient $R_he(P)$ still depends on h. This can partially be cured by using the interior L^∞-error estimate (6.4.40) for the function e,

$$\|e - R_he\|_{L^\infty(\Omega_h'')} \le ch^2L(h)\big\{\|e\|_{H^{2,\infty}(\Omega_h')} + \|e\|_{H^2(\Omega)}\big\}, \tag{6.4.72}$$

which holds on any fixed interior mesh-subregions $\Omega_h'' \subset \Omega_h' \subset \overline{\Omega}$ (unions of triangles), which have increasing positive distance to the corner points of the polygonal domain Ω (see Fig. 6.6). Notice that (in view of the convexity of Ω) the regularity $e \in H^2(\Omega)$ is always guaranteed for $\Delta e \in L^2(\Omega)$, i. e., $u \in H^4(\Omega)$. The higher degree of regularity $e \in H^{2,\infty}(\Omega')$ on interior subdomains $\Omega' \subset \Omega$ requires $\Delta e \in C^\alpha(\Omega)$ for some $\alpha > 0$, i. e., $u \in C^{4+\alpha}(\Omega)$. Now, combining the preliminary error expansion (6.4.71) with the error estimate (6.4.72), we obtain the final result

$$R_hu(P) = u(P) + h^2e(P) + \mathcal{O}(h^4L(h)),$$

at interior nodal points $P \in \Omega''$. The refined global estimate of the remainder term depending on the maximal interior angle ω of $\partial\Omega$ follows in the same way using the global L^∞-error estimate

$$\|e - R_he\|_{L^\infty} \le ch^{\min(2,1+\pi/\omega)};$$

for the very technical proof, we refer to the literature. However, there is a simple argument, which gives at least a sub-optimal result. Observing that on the convex domain Ω, the coefficiant e has H^2 regularity, we can use the lower-order L^∞-error estimate (exercise)

$$\|e - R_he\|_{L^\infty} \le ch\|e\|_{H^2},$$

to obtain the reduced order $\mathcal{O}(h^3)$ for the remainder term. Q.E.D.

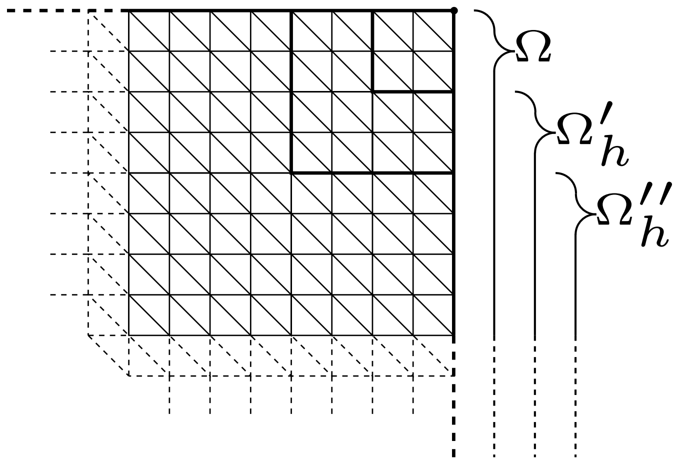

Figure 6.5: Interior mesh-subregions $\Omega_h'' \subset \Omega_h' \subset \overline{\Omega}$ having increasing positive distance to the corner points of $\partial\Omega$.

6.4.4 Proofs of L^∞-stability and error estimates

In the following, we give a proof of the stability estimate (6.4.69) for the Ritz projection $R_h : H_0^1(\Omega) \cup C_0(\bar{\Omega}) \to V_h$. For this the mesh $\mathbb{T}_h$ does not need to be three-directional, only quasi-uniformity is required as we want to use the "inverse relation" for finite elements

$$\|\nabla \varphi_h\|_T \le ch^{-1}\|\varphi_h\|_T, \quad \varphi_h \in V_h,\ T \in \mathbb{T}_h, \tag{6.4.73}$$

with a constant independent of h. We note that even this condition can be relaxed in various directions (e. g., Section 3.1 of the Lecture Notes "Special Topics in Numerics I (FEM for Nonlinear Problems)").

Let $z \in \Omega$ be an arbitrary but fixed point and $T_z \in \mathbb{T}_h$ a triangle containing z. There exists a function δ_z^h (regularized Dirac function) with support $\operatorname{supp}(\delta_z^h) \subset T_z$ and $\|\delta_z^h\|_{L^\infty} \le ch^{-2}$ such that (exercise)

$$(\varphi_h, \delta_z^h) = \varphi_h(z), \quad \varphi_h \in V_h. \tag{6.4.74}$$

We define a corresponding "regularized" Green's function $g_z^h \in H_0^1(\Omega) \cap H^2(\Omega)$ as the solution of the problem

$$(\nabla\varphi, \nabla g_z^h) = (\varphi, \delta_z^h) \quad \forall \varphi \in H_0^1(\Omega). \tag{6.4.75}$$

For this, we have the L^1-error estimate

$$\|\nabla(g_z^h - R_h g_z^h)\|_{L^1} + h\|\nabla^2 g_z^h\|_{L^1} \le chL(h), \tag{6.4.76}$$

and the L^2-error estimate

$$\|g_z^h - R_h g_z^h\|_{L^2} \le chL(h)^{1/2}, \tag{6.4.77}$$

the proofs of which will be supplied below. Using equations (6.4.74) and (6.4.75), we conclude for functions $v \in H_0^1(\Omega) \cap C_0(\partial\mathbb{T}_h)$ that

$$R_h v(z) = (\nabla R_h v, \nabla g_z^h) = \sum_{T\in\mathbb{T}_h} \int_{\partial T} v\,\partial_n R_h g_z^h\,ds = \sum_{T\in\mathbb{T}_h} \int_{\partial T} v\,\partial_n (R_h g_z^h - g_z^h)\,ds\,.$$

Employing the cellwise trace inequality in $H^{1,1}(T)$ (exercise),

$$\int_{\partial T} |\partial_n (R_h g_z^h - g_z^h|\,ds \le c\big\{h^{-1}\|\nabla(R_h g_z^h - g_z^h)\|_{L^1(T)} + \|\nabla^2 g_z^h\|_{L^1(T)}\big\}, \quad T\in\mathbb{T}_h,$$

it follows that

$$|R_h v(z)| \le c\|v\|_{C(\partial\mathbb{T}_h)}\big\{h^{-1}\|\nabla(R_h g_z^h - g_z^h)\|_{L^1} + \|\nabla^2 g_z^h\|_{L^1}\big\}.$$

This together with the error estimate (6.4.76) proves the desired stability estimate (6.4.69), since the point z was chosen arbitrarily in Ω.

Stability and error estimates for the standard Ritz projection

As a by-product of the above argument, we also obtain a stability result for the normal Ritz projection.

Lemma 6.4: *For functions $v \in H_0^1(\Omega) \cap H^{1,\infty}(\Omega)$ there holds*

$$\|R_h v\|_{L^\infty} \le c\big\{\|v\|_{L^\infty} + hL(h)\|\nabla v\|_{L^\infty}\big\}. \tag{6.4.78}$$

Proof: Using the foregoing notation for functions it follows that

$$\begin{aligned} R_h v(z) &= (\nabla R_h v, \nabla g_z^h) = (\nabla R_h v, \nabla R_h g_z^h) \\ &= (\nabla v, \nabla R_h g_z^h) = (\nabla v, \nabla(R_h g_z^h - g_z^h)) + (\nabla v, \nabla g_z^h) \\ &= (\nabla v, \nabla(R_h g_z^h - g_z^h)) + (v, \delta_z^h), \end{aligned}$$

and, consequently,

$$|R_h v(z)| \le c\|\nabla v\|_{L^\infty}\|\nabla(R_h g_z^h - g_z^h)\|_{L^1} + c\|v\|_{L^\infty(T_z)}.$$

Then, using again the L^1-error estimate (6.4.76) for the Green's function, we obtain the asserted L^∞-stability estimate. Q.E.D.

Applying the stability estimate (6.4.78) to the function $v := u - I_h u$, we obtain

$$\|R_h(u - I_h u)\|_{L^\infty} \le c\big\{\|u - I_h u\|_{L^\infty} + hL(h)\|\nabla(u - I_h u)\|_{L^\infty}\big\},$$

and finally

$$\|R_h u - u\|_{L^\infty} \le c\big\{\|u - I_h u\|_{L^\infty} + hL(h)\|\nabla(u - I_h u)\|_{L^\infty}\big\}.$$

In view of the usual interpolation estimate for "linear" finite elements,

$$\|u - I_h u\|_{L^\infty} + h\|\nabla(u - I_h u)\|_{L^\infty} \le ch^2\|\nabla^2 u\|_{L^\infty},$$

we obtain the L^∞-error estimate

$$\|R_h u - u\|_{L^\infty} \le ch^2 L(h)\|\nabla^2 u\|_{L^\infty}, \tag{6.4.79}$$

provided that the solution $u \in H_0^1(\Omega)$ has global $H^{2,\infty}$-regularity. However, this assumption is unrealistic in general even on convex polygonal domains. On convex polygonal domains, we generally only have $u \in H^2(\Omega)$ but for smooth right-hand side f higher regularity holds in the interior of Ω. More precisely in subdomains $\Omega' \subset \Omega$ having positive distance from the corner points of $\partial\Omega$. This leads to the question for "interior" L^∞-error estimates. In the following, for technical reasons, we consider "mesh-subregions" $\Omega_h'' \subset \Omega_h' \subset \overline{\Omega}$, which are unions of triangles and have increasing distances, uniformly w.r.t. h, to the corner points of $\partial\Omega$ (see Fig. 6.6).

Lemma 6.5: *Let $\Omega_h'' \subset \Omega_h' \subset \overline{\Omega}$ be mesh-subregions having increasing positive distance to the corner points of $\partial\Omega$. Then, for any function $v \in H_0^1(\Omega) \cap H^{1,\infty}(\Omega_h')$ there holds the "interior" stability estimate*

$$\|R_h v\|_{L^\infty(\Omega_h'')} \le c\|v\|_{L^\infty(\Omega_h')} + chL(h)\big\{\|\nabla v\|_{L^\infty(\Omega_h')} + \|\nabla v\|_{L^2}\big\}. \tag{6.4.80}$$

Proof: Recalling the foregoing notation, for a function $v \in H_0^1(\Omega) \cap H^{1,\infty}(\Omega_h')$ there holds at any fixed point $z \in T_z \subset \Omega_h''$:

$$\begin{aligned} R_h v(z) &= (\nabla v, \nabla(R_h g_z^h - g_z^h)) + (v, \delta_z^h) \\ &= (\nabla v, \nabla(R_h g_z^h - g_z^h))_{\Omega_h'} + (\nabla v, \nabla(R_h g_z^h - g_z^h))_{\Omega\setminus\Omega_h'} + (v, \delta_z^h), \end{aligned}$$

and, consequently,

$$\begin{aligned} |R_h v(z)| \le \|\nabla v\|_{L^\infty(\Omega')}\|\nabla(R_h g_z^h - g_z^h)\|_{L^1(\Omega')} + \|\nabla v\|_{L^2(\Omega\setminus\Omega'}\|\nabla(R_h g_z^h - g_z^h)\|_{L^2(\Omega\setminus\Omega_h')} \\ + c\|v\|_{L^\infty(T_z)}. \end{aligned}$$

Below, we will provide the interior error estimate

$$\|\nabla(R_h g_z^h - g_z^h)\|_{L^2(\Omega\setminus\Omega_h')} \le chL(h)^{1/2}, \tag{6.4.81}$$

where the constant c depends on $\mathrm{dist}(\Omega \setminus \Omega_h', z)$. Using this L^2-error estimate and the L^1-error estimate (6.4.76), we conclude that

$$|R_h v(z)| \le chL(h)\|\nabla v\|_{L^\infty(\Omega')} + hL(h)^{1/2}\|\nabla v\|_{L^2(\Omega\setminus\Omega')} + \|v\|_{L^\infty(T_z)}. \tag{6.4.82}$$

This implies the asserted stability estimate. Q.E.D.

Now let the solution $u \in H_0^1(\Omega)$ of problem (6.4.36) possess the (realistic) regularity $u \in H^2(\Omega) \cap H^{2,\infty}(\Omega'_h)$ in an interior mesh-subregion $\Omega'_h \subset \overline{\Omega}$ having positive distance to the corner points of $\partial\Omega$. Then, using again the stability estimate (6.4.80) for the function $v := u - I_h u$, we obtain

$$\|R_h(u-I_hu)\|_{L^\infty(\Omega''_h)} \le c\big\{\|u-I_hu\|_{L^\infty(\Omega'_h)}+hL(h)\|\nabla(u-I_hu)\|_{L^\infty(\Omega'_h)}+h\|\nabla(u-I_hu)\|_{L^2}\big\},$$

and, consequently, analogously as above,

$$\|R_hu - u\|_{L^\infty(\Omega''_h)} \le c\big\{h^2L(h)\|u\|_{H^{2,\infty}(\Omega'_h)} + h^2\|u\|_{H^2}\big\}. \tag{6.4.83}$$

This is the desired interior L^∞-error estimate (6.4.72).

6.4.5 Further results on L^∞-error behavior

The major weak point of the error expansion result of Theorem 6.7 is the unrealistic assumption of $C^{4+\alpha}$-regularity of the solution u in $\overline{\Omega}$ up to the corner points of $\partial\Omega$. This is satisfied only in very exceptional situations, e. g., on the unit square $\Omega = (0,1)^2$ for the special right-hand side $f(x,y) = 2\pi^2\sin(\pi x)\sin(\pi y)$, with the corresponding smooth solution $u(x,y) = \sin(\pi x)\sin(\pi y)$ (eigenfunction of the Laplace operator in $H_0^1(\Omega)$). In general, the weak solution $u \in H_0^1(\Omega)$ of the Poisson problem (6.4.36) (and likewise also of more general elliptic operators) show a specific singular behavior at the corner points of Ω (see Grisvard [8])). For that reason interior expansion results similar to the interior L^∞-error estimate (6.4.83) beyond the order $\mathcal{O}(h^2)$ do not hold for solutions with realistic global regularity less than $C^{2+\alpha}(\overline{\Omega})$.

Let $z_j,\ j = 1,\dots,J$, be the corner points of Ω with interior angles $\omega_j \in (0,\pi)$, and let (r_j,θ_j) denote the corresponding local polar coordinates at z_j. Then, the weak solution $u \in H_0^1(\Omega)$ possesses a representation of the form

$$u = \sum_j \gamma_j s_j + \tilde{u}, \tag{6.4.84}$$

where

$$s_j(r_j,\theta_j) = r_j^{\pi/\omega_j}\sin\big(\pi/\omega_j\,\theta_j\big), \quad \text{for } \omega_j \ne \pi/2$$

and $\tilde{u} \in C^{2,\varepsilon}(\overline{\Omega})$, for some $\varepsilon > 0$. In the special case $\omega_j = \pi/2$, one has

$$s_j(r_j,\theta_j) = r_j^2\big(\ln(r_j)\sin(2\theta_j) + \theta_j\cos(2\theta_j) + \tfrac{1}{2}\pi\sin^2(\theta_j)\big).$$

From this representation, we see that on a convex polygonal domain there always holds $u \in C^{1+\alpha}(\overline{\Omega})$ for some $\alpha > 0$. In this case, we have the following L^∞-error estimates:

$$\|u - R_hu\|_{L^\infty} = \begin{cases} \mathcal{O}(h^{1+\alpha}), & \text{for } 0 \le \alpha < 1, \\ \mathcal{O}(h^2L(h)), & \text{for } \alpha = 1. \end{cases} \tag{6.4.85}$$

On uniform meshes, for $u \in C^{2+\alpha}(\overline{\omega})$ with some $\alpha > 0$, we have the logarithm-free estimate

$$\|u - R_h u\|_{L^\infty} = \mathcal{O}(h^2), \tag{6.4.86}$$

while in general for the weak solution $u \in H_0^1(\Omega)$ this only holds in subdomains $\Omega' \subset \Omega$ having positive distance to the corner points of $\partial\Omega$,

$$\|u - R_h u\|_{L^\infty(\Omega')} = \mathcal{O}(h^2). \tag{6.4.87}$$

A further consequence of our expansion results is the following "super-approximation" estimate for linear finite elements. If $u \in C^{4+\alpha}(\bar{\Omega})$, with some $\alpha > 0$, on a uniform mesh consisting of equilateral triangles there holds (exercise)

$$\|u - R_h u\|_{L^\infty} = \mathcal{O}(h^4). \tag{6.4.88}$$

6.4.6 Estimates for the regularized Green's function

In the following, we prove error estimates for the regularized Green's function $g_z^h \in H_0^1(\Omega)$: the L^1-error estimate (6.4.76), the L^2-error estimate (6.4.77), the local error estimate (6.4.81), and some furser error estimates for later use.

We begin with some technical preparations. We define the weight function

$$\sigma(x) := (|x - z|^2 + h^2)^{1/2}.$$

By elementary calculation, we see that

$$\sigma \geq h, \quad |\nabla \sigma| \leq 1, \quad \|\sigma^{-1}\| \leq c\, L(h)^{1/2}, \tag{6.4.89}$$

where again $L(h) = (|\ln(h)| + 1$. Further, for each mesh cell $T \in \mathbb{T}_h$,

$$\max_T \sigma \leq \min_T \sigma + h_T \max_T |\nabla \sigma| \leq c \min_T \sigma. \tag{6.4.90}$$

(Remark: If the mesh $\mathbb{T}_h$ is not size-regular, then there holds $\max_T \sigma \leq \min_T \sigma + h_T$.) We recall the standard cellwise interpolation estimate

$$\|v - I_h v\|_T + h_T \|\nabla(v - I_h v)\|_T \leq c h_T^2 \|\nabla^2 v\|_T, \quad T \in \mathbb{T}_h,\ v \in H^2(T). \tag{6.4.91}$$

and the L^2-error estimate

$$\|v - R_h v\| + h\|\nabla(v - R_h v)\| \leq c h^2 \|\nabla^2 v\|, \quad v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{6.4.92}$$

The assertion of the following auxiliary lemma is crucial for the argument used below..

Lemma 6.6: *There hold the following a priori estimates:*

$$\|g_z^h\|_\infty \le cL(h), \tag{6.4.93}$$
$$\|\nabla g_z^h\| + \|\sigma\nabla^2 g_z^h\| \le cL(h)^{1/2}, \tag{6.4.94}$$
$$\|\nabla^2 g_z^h\|_{L^1} \le cL(h). \tag{6.4.95}$$

Proof: i) The true Green's function G_x on Ω corresponding to an arbitrary point $x \in \Omega$ admits the well-known estimate

$$|G_x(y)| \le c\{|\ln(|y-x|)| + 1\},$$

which may be derived by using the maximum principle. Then, from the estimate (exercise)

$$|g_z^h(x)| = |(\nabla g_z^h, \nabla G_x)| = |(\delta_z^h, G_x)| \le |T_z|^{-1}\int_{T_z} |G_x|\, dy \le cL(h),$$

we obtain (6.4.93).

ii) Observing that

$$\|\nabla g_z^h\|^2 = (\delta_z^h, g_z^h) \le \|g_z^h\|_\infty \le cL(h),$$

we obtain the first part of (6.4.94). Further, by the usual H^2 a priori estimate, we obtain (6.4.95),

$$\|\nabla^2 g_z^h\| \le c\|\Delta g_z^h\| = c\|\delta_z^h\| \le ch^{-1}. \tag{6.4.96}$$

Next, we set $\xi := x - z$ and find

$$|\xi_i\nabla^2 g_z^h| \le |\nabla^2(\xi_i g_z^h)| + |\nabla g_z^h|,$$

and consequently,

$$\begin{aligned}\|\sigma\nabla^2 g_z^h\|^2 &= \sum_{i=1}^2 \|\xi_i\nabla^2 g_z^h\|^2 + h^2\|\nabla^2 g_z^h\|^2 \\ &\le c\sum_{i=1}^2 \big\{\|\nabla^2(\xi_i g_z^h)\|^2 + \|\nabla g_z^h\|^2\big\} + h^2\|\nabla^2 g_z^h\|^2.\end{aligned}$$

By the usual H^2 a priori estimate,

$$\begin{aligned}\|\nabla^2(\xi_i g_z^h)\| &\le \|\Delta(\xi_i g_z^h)\| \le \|\xi_i\Delta g_z^h\| + \|\nabla g_z^h\| \\ &= \|\xi_i\delta_z^h\| + \|\nabla g_z^h\| \le c + cL(h)^{1/2}.\end{aligned}$$

Combining the foregoing estimates, we obtain the second part of (6.4.94),

$$\|\sigma\nabla^2 g_z^h\| \le cL(h)^{1/2}.$$

iii) Finally, we estimate

$$\|\nabla^2 g_z^h\|_{L^1} \le \|\sigma^{-1}\|\|\sigma\nabla^2 g_z^h\| \le cL(h),$$

to obtain (6.4.95). Q.E.D.

Lemma 6.7: *For the regularized Green's function* g_z^h *there holds the* L^1*-error estimate*

$$\|\nabla(g_z^h - R_h g_z^h)\|_{L^1} \le chL(h), \tag{6.4.97}$$

and the L^2*-error estimate*

$$\|g_z^h - R_h g_z^h\| \le ch. \tag{6.4.98}$$

Proof: i) We set $\eta := g_z^h - R_h g_z^h$. Combining the L^2-error estimate (6.4.92) for $v := g_z^h$ with the a priori estimate (6.4.96), we have

$$\|\eta\| + h\|\nabla\eta\| \le ch^2\|\nabla^2 g_z^h\| \le ch, \tag{6.4.99}$$

which proves (6.4.98). (Remark: If the mesh $\mathbb{T}_h$ is not size-regular, one has to avoid the occurrence of the norm $\|\nabla^2 g_z^h\| \approx h_{T_z}^{-1}$. This can be circumvented by using the estimates $\|\eta\| \le ch\|\nabla g_z^h\|$ and $\|\nabla g_z^h\| \le cL(h_{T_z})^{1/2}$.)

ii) Further there holds

$$\|\nabla\eta\|_1 \le \|\sigma^{-1}\|\,\|\sigma\nabla\eta\| \le cL(h)^{1/2}\|\sigma\nabla\eta\|. \tag{6.4.100}$$

For the term on the right, we have

$$\|\sigma\nabla\eta\|^2 = (\nabla\eta, \nabla(\sigma^2\eta) - (\nabla\eta, \eta\nabla\sigma^2) =: E_1 - E_2.$$

The terms E_1 and E_2 will be estimated separately. First, using Galerkin orthogonality, we obtain

$$E_1 = (\nabla\eta, \nabla(\sigma^2\eta - \psi_h))$$

with the nodal interpolant $\psi_h := I_h(\sigma^2\eta) \in V_h$. This term is estimated further using the cellwise interpolation estimate (6.4.91),

$$E_1 \le \sum_{T\in\mathbb{T}_h} \|\nabla\eta\|_T\|\nabla(\sigma^2\eta - \psi_h)\|_T \le c\sum_{T\in\mathbb{T}_h} h_T\|\nabla\eta\|_T\|\nabla^2(\sigma^2\eta)\|_T.$$

For the second factors on the right, we have

$$\|\nabla^2(\sigma^2\eta)\|_T \le c\big\{\|\eta\|_T + \|\sigma\nabla\eta)\|_T + \|\sigma^2\nabla^2 g_z^h\|_T\big\},$$

and consequently,

$$E_1 \le c\sum_{T\in\mathbb{T}_h} h_T\Big\{\|\nabla\eta\|_T\big\{\|\eta\|_T + \|\sigma\nabla\eta\|_T\big\} + \|\nabla\eta\|_T\|\sigma^2\nabla^2 g_z^h\|_T\Big\}.$$

In view of the relation $\max_T \sigma \le c \min_T \sigma$, we have (Remark: At this point the argument needs to be modified if the mesh $\mathbb{T}_h$ is not size-regular.)

$$\|\nabla\eta\|_T \|\sigma^2\nabla^2 g_z^h\|_T \le c\|\sigma\nabla\eta\|_T \|\sigma\nabla^2 g_z^h\|_T.$$

Hence, by Schwarz's inequality,

$$E_1 \le ch\|\nabla\eta\|\big\{\|\eta\| + \|\sigma\nabla\eta\|\big\} + ch\|\sigma\nabla\eta\|\|\sigma\nabla^2 g\|.$$

In view of the L^2-error estimate (6.4.99) and the a priori estimate (6.4.94), we conclude

$$E_1 \le \tfrac{1}{4}\|\sigma\nabla\eta\|^2 + ch^2 L(h).$$

For the second term E_2, we analogously obtain

$$E_2 \le c\|\sigma\nabla\eta\|\|\eta\| \le \tfrac{1}{4}\|\sigma\nabla\eta\|^2 + ch^2.$$

Combining the estimates for E_1 and E_2, we obtain

$$\|\sigma\nabla\eta\|^2 \le \tfrac{1}{2}\|\sigma\nabla\eta\|^2 + ch^2 L(h).$$

and eventually

$$\|\sigma\nabla\eta\| \le chL(h)^{1/2}, \tag{6.4.101}$$

which in view of (6.4.100) yields the desired L^1-error estimate (6.4.97). Q.E.D.

As a simple by-product of the argument used in the proof of the foregoing lemma, we also obtain the local error estimate (6.4.81).

Lemma 6.8: *Let $\Omega'_z \subset \Omega$ be a subdomain having a fixed positive distance to the point z. Then there holds*

$$\|\nabla(R_h g_z^h - g_z^h)\|_{\Omega'_z} \le chL(h)^{1/2}, \tag{6.4.102}$$

where the constant c depends on $\operatorname{dist}(z, \Omega'_z)$.

Proof: Since z has positive distance from Ω'_z, we can estimate

$$\|\nabla(R_h g_z^h - g_z^h)\|_{\Omega'_z} \le c\|\sigma\nabla(R_h g_z^h - g_z^h)\|.$$

Then, the asserted estimate follows from the weighted error estimate (6.4.101). Q.E.D.

Remark 6.6: The above error estimates for the regularized Green's function have been proven for mesh families being uniformly size- and shape-regular. However, below, in approximating curved boundaries or in resolving local irregularities one may create meshes which are only weakly size-regular ("'polynomial size-regular"' in the sense that

$$\min_{T\in\mathbb{T}_h} h_T \approx \max_{T\in\mathbb{T}_h} h_T^\alpha,$$

with some fixed parameter $\alpha \geq 1$ (usually $\alpha \in [1,2]$). By modifying the above proof and observing that $L(h_{\min}) \approx \alpha L(h)$ it can be shown that these error estimates remain essentially true also on such meshes (exercise).

6.4.7 Error expansion on blockwise uniform meshes

The requirement on the mesh to be three-directional is rather restrictive and limits the applicability of the error expansion result of Theorem 6.7 to very special situations. Therefore, in the next step, we extend this result to triangulations, which are only "‘blockwise"’ three-directional. Let the polygonal domain Ω be subdevided into a finitely many "macro-triangles" $\Omega^{(j)}$ and let $\mathbb{T}_h^{(j)}$ be be uniform (three-directional) triangulations of the regions $\bar{\Omega}^{(j)}$ such that $\mathbb{T}_h := \cup_j \mathbb{T}_h^{(j)}$ is a regular triangulation of $\bar{\Omega}$.

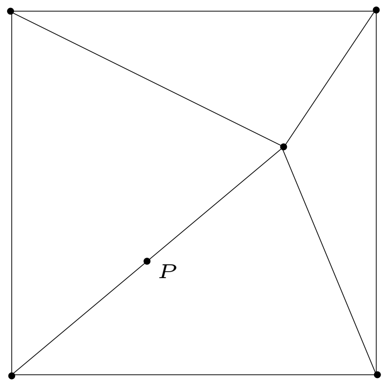

Figure 6.6: Macrodicomposition of $\bar{\Omega}$.

Theorem 6.8: *Let $u \in H_0^1(\Omega) \cap C^{4+\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. Then, on a blockwise three-directional mesh $\mathbb{T}_h := \cup_j \mathbb{T}_h^{(j)}$ for the Ritz-projection there holds the error expansion*

$$R_h u(P) = u(P) + h^2 e(P) + \mathcal{O}(h^4 L(h)), \tag{6.4.103}$$

at interior nodel points P having positive distance to the corner points of $\partial\Omega$ and to the vertices of the macro-triangulation. The expansion also holds at nodal points on the macro-edges, though the triangulation is not uniform there.

Proof: We employ a slightly different argument then that in the proof of Theorem 6.7, which directly uses local estimates for the regularized Green's function g_z^h defined in Section 6.4.4 rather than global stability estimates. In thie way, we can derive more localized expansion results.

i) With the notation from above, $z \in T_z \in \mathbb{T}_h$ and δ_z^h, we have by Galerkin orthogonality:

$$\begin{aligned}(R_h u - I_h u)(z) &= (R_h u - I_h u, \delta_z^h) = (\nabla(R_h u - I_h u), \nabla g_z^h)\\ &= (\nabla(R_h u - I_h u), \nabla R_h g_z^h) = (\nabla(u - I_h u), \nabla R_h g_z^h).\end{aligned}$$

Hence, at nodal points $z=P$ there holds

$$(R_h u - u)(P) = (\nabla(u - I_h u), \nabla R_h g_P^h). \tag{6.4.104}$$

ii) Next, setting $\varphi_h := R_h g_P^z$, we recall the representation (6.4.57) for the cellwise consistancy error derived above:

$$(\nabla(u - I_h u), \nabla \varphi_h) = \sum_j \sum_{T\in\mathbb{T}_h^{(j)}} \sum_{i=1}^3 I_T^i, \tag{6.4.105}$$

where

$$\begin{aligned} I_T^i = {} & h^2\beta_1 \frac{\lambda_i^4}{2\alpha} \int_{S_{i+2}} D_{i+2}\varphi_h D_i^2 u\, ds + h^2\beta_1 \frac{\lambda_i^4 h_{i+1}\lambda_{i+2}}{4\alpha^2} \int_T \varphi_h D_{i+2} D_{i+1} D_i^2 u\, dx \\ & - h^2\beta_1 \frac{\lambda_i^3\lambda_{i+1}}{2\alpha} \int_{S_i} \varphi_h D_{i+1} D_i^2 u\, ds + h^2\beta_1 \frac{\lambda_i^4}{2\alpha} \int_{S_{i+1}} \varphi_h D_{i+1} D_i^2 u\, ds \\ & - h^2\beta_1 \frac{\lambda_i^3\lambda_{i+2}}{2\alpha} \int_{S_i} n^i \cdot n^{i+2} D_i\varphi_h D_i^2 u\, ds + h^4\lambda_i^4 \int_{S_i} \beta_2^i D_i^4 u\, \partial_{n^i}\varphi_h\, ds\,. \end{aligned} \tag{6.4.106}$$

This representation holds on each single triangle $T \in \mathbb{T}_h$. Now, summing these terms separately for all cells $T \in \mathbb{T}_h^{(j)}$, we obtain the following:

a) The line integrals of the type

$$\int_{S_i} D_i\varphi_h \,\dots\, ds, \qquad \int_{S_i} \varphi_h \,\dots\, ds$$

over interior sides in $\Omega^{(j)}$ are cancelled. The contributions from sides $S_i \subset \partial\Omega^{(j)}$ add up:

$$\begin{aligned} h^2 L_h^{(j)}(u,\varphi_h) = h^2 \frac{\beta_1}{2\alpha} \sum_{i=1}^3 \Big\{ & \lambda_{i+1}^4 \int_{\Gamma_i} D_i\varphi_h D_{i+1}^2 u\, ds - \lambda_i^3\lambda_{i+2} n\cdot n^{i+2} \int_{\Gamma_i} D_i\varphi_h D_i^2 u\, ds \\ & - \lambda_i^3\lambda_{i+1} \int_{\Gamma_i} \varphi_h D_{i+1} D_i^2 u\, ds + \lambda_{i+2}^4 \int_{\Gamma_i} \varphi_h D_i D_{i+2}^2 u\, ds \Big\}, \end{aligned}$$

where $\lambda_i = \lambda_i^{(j)}$, $\alpha = \alpha^{(j)}$ and $D_i = D_i^{(j)}$ are the characteristic quantities of the triangulation $\mathbb{T}_h^{(j)}$ and $\Gamma_i = \Gamma_i^{(j)}$ is the part of $\partial\Omega^{(j)}$ in the direction $t_i^{(j)}$.

b) The area integrals and the remainder term combine to

$$h^2 e_h^{(j)}(u,\varphi_h) := h^2\beta_1 \frac{\lambda_1\lambda_2\lambda_3}{4\alpha^2} \int_{\Omega^{(j)}} \varphi_h \Big(D_1 D_2 D_3 \sum_{i=1}^3 \lambda_i^3 D_i \Big) u\, dx, \tag{6.4.107}$$

$$h^4 \rho_h^{(j)}(u,\varphi_h) := h^4 \sum_{T\in\mathbb{T}_h^{(j)}} \int_{\partial T} D_h^{(4)}(u)\partial_n\varphi_h\, ds\,, \tag{6.4.108}$$

where the function $D_h^{(4)}(u) \in C_0(\partial\mathbb{T}_h^{(j)})$ is defined on the set of sides $\partial\mathbb{T}_h^{(j)}$ by

$$D_h^{(4)}(u)|_{S_i} := \lambda_i^4 \beta_2^i D_i^4 u,$$

and satisfies

$$D_h^{(4)}(u)|_{\partial\Omega} = 0, \qquad \|D_h^{(4)}(u)\|_{C(\partial\mathbb{T}_h)} \le c\|u\|_{C^4}. \tag{6.4.109}$$

With this notation, we define

$$\begin{aligned} e_h(u,\varphi_h) &:= \sum_j \big\{ e_h^{(j)}(u,\varphi_h) + L_h^{(j)}(u,\varphi_h) \big\}, \\ \rho_h(u,\varphi_h) &:= \sum_j \rho_h^{(j)}(u,\varphi_h). \end{aligned}$$

This leads us again to the following error representation for the consistency error in nodal points P:

$$(\nabla(u - I_h u), \nabla\varphi_h) = h^2 e_h(u,\varphi_h) + h^4 \rho_h(u,\varphi_h),$$

and consequently, recalling $\varphi_h = R_h g_P^h$,

$$R_h u(P) = u(P) + h^2 e_h(u, R_h g_P^h) + h^4 \rho_h(u, R_h g_P^h). \tag{6.4.110}$$

iii) The remainder term $\rho_h(u, R_h g_P^h)$ has the same form as in the case of a globally uniform mesh. It can be treated again using the cellwise trace inequality in $H^{1,1}(T)$ as follows:

$$\begin{aligned} |\rho_h(u, R_h g_P^h)| &= \Big| \sum_j \sum_{T\in\mathbb{T}^{(j)}} \int_{\partial T} D_h^{(4)}(u)\partial_n R_h g_P^h \, ds \Big| \\ &= \Big| \sum_{T\in\mathbb{T}} \int_{\partial T} D_h^{(4)}(u)\partial_n (R_h g_P^h - g_P^h)\, ds \Big| \\ &\le c\|u\|_{C^4} \sum_{T\in\mathbb{T}} \int_{\partial T} |\partial_n (R_h g_P^h - g_P^h)|\, ds \\ &\le c\|u\|_{C^4} \sum_{T\in\mathbb{T}} \Big\{ h^{-1} \int_T |\nabla (R_h g_P^h - g_P^h)\, dx + \int_T |\nabla^2 g_P^h)|\, dx \Big\} \\ &\le c\|u\|_{C^4} \big\{ h^{-1} \|\nabla (R_h g_P^h - g_P^h)\|_{L^1} + \|\nabla^2 g_P^h\|_{L^1} \big\}. \end{aligned}$$

Then, the already proven L^1 estimates for the regularized Green's function imply that

$$|\rho_h(u, R_h g_P^h)| = \mathcal{O}(L(h)). \tag{6.4.111}$$

holds uniformly for all nodal points $P \in \Omega$.

iv) Now, we have to replace the mesh-dependent term $e_h(u, R_h g_P^h)$ by a mesh-independent term plus a term of the correct order $\mathcal{O}(h^2 L(h))$. To this end, we insert the exact Greens's function g_P . Notice that in view of the relation

$$(\nabla R_h g_P^h, \nabla\varphi_h) = (\nabla g_P^h, \nabla\varphi_h) = \varphi_h(P) \\ = (\nabla g_P, \nabla\varphi_h) = (\nabla R_h g_P, \nabla\varphi_h), \quad \varphi_h \in V_h,$$

$R_h g_P^h$ is also the Ritz projection of g_P (in an extended sense): $R_h g_P^h = R_h g_P$. Setting $e(P) := e_h(u, g_P)$, we obtain the intermediate result

$$R_h u(P) = u(P) + h^2 e(P) + h^2 e_h(u, R_h g_P - g_P) + \mathcal{O}(h^4 L(h)). \tag{6.4.112}$$

It remains to estimate

$$e_h(u, R_h g_P - g_P) = \sum_j \big\{ e_h^{(j)}(u, R_h g_P - g_P) + L_h^{(j)}(u, R_h g_P^h - g_P) \big\}.$$

We begin with the first term in the sum. Let $\hat{e} \in H_0^1(\Omega)$ be the solution of the problem

$$(\nabla\hat{e}, \nabla\varphi) = \sum_j e_h^{(j)}(u, \varphi) \quad \forall\varphi \in H_0^1(\Omega), \tag{6.4.113}$$

i. e., the weak solution of the boundary value problem

$$-\Delta\hat{e} = \beta_1 \frac{\lambda_1\lambda_2\lambda_3}{4\alpha^2} \Big(D_1 D_2 D_3 \sum_{i=1}^3 \lambda_i^3 D_i \Big) u \quad \text{in } \Omega, \quad \hat{e} = 0 \ \text{ on } \partial\Omega, \tag{6.4.114}$$

where the right-hand side is defined piecewise with respect to the macro-triangulation $\{\Omega^{(j)}\}_j$, i. e., it is discontinuous since the directional derivatives D_i and the other parameters are different in each of the $\Omega^{(j)}$. Since $u \in C^{4+\alpha}(\bar{\Omega})$ this right-hand side is in $C^\alpha(\bar{\Omega}^{(j)})$ on each of the macro-triangles. Therefore, the solution $\hat{e} \in H_0^1(\Omega)$ is globally in $H^2(\Omega)$ and locally in $C^{2+\alpha}(\bar{\Omega}^{(j)} \setminus B^{(j)})$, where $B^{(j)}$ are fixed neighborhoods of the vertices of the macro-triangle $\Omega^{(j)}$. Since the global triangulation $\mathbb{T}_h$ matches the macro-edges, we have the L^∞-error estimate

$$|(\hat{e} - R_h\hat{e})(z)| = \mathcal{O}(h^2 L(h)), \tag{6.4.115}$$

at points $z \in \bar{\Omega}$ uniformly bounded away from the corner points of $\partial\Omega$ and from the vertices of the macro-triangulation. This implies using Galerkin orthogonality and the definition of g_P that

$$\sum_j e_h^{(j)}(u, R_h g_P - g_P) = (\nabla e, \nabla(R_h g_P - g_P)) = (\nabla(\hat{e} - R_h\hat{e}), \nabla(R_h g_P - g_P)) \\ = -(\nabla(\hat{e} - R_h\hat{e}), \nabla g_P) = (R_h\hat{e} - \hat{e})(P).$$

Consequently, in view of the pointwise error estimate (6.4.115):

$$\Big|\sum_j e_h^{(j)}(u, R_h g_P - g_P)\Big| = \mathcal{O}(h^2 L(h)), \tag{6.4.116}$$

at nodal points $P \in \bar{\Omega}$ uniformly bounded away from the corner points of $\partial\Omega$ and from the vertices of the macro-triangulation.

v) Next, we consider the integrals along the macro-edges:

$$\sum_j L_h^{(j)}(u, R_h g_P - g_P)$$

where, recalling the representation from above,

$$\begin{aligned} L_h^{(j)}(u,\varphi) = \frac{\beta_1}{2\alpha}\sum_{i=1}^{3}\Big\{\lambda_{i+1}^4 \int_{\Gamma_i} D_i\varphi D_{i+1}^2 u\, ds - \lambda_i^3\lambda_{i+2} n\cdot n^{i+2}\int_{\Gamma_i} D_i\varphi D_i^2 u\, ds \\ - \lambda_i^3\lambda_{i+1}\int_{\Gamma_i}\varphi D_{i+1}D_i^2 u\, ds + \lambda_{i+2}^4\int_{\Gamma_i}\varphi D_i D_{i+2}^2 u\, ds\Big\}, \end{aligned}$$

We have to estimate integrals over edges Γ_i of the macro-triangulation of the form

$$\int_{\Gamma_i} D_i(R_h g_P - g_P) D_{i+1}^2 u\, ds, \qquad \int_{\Gamma_i}(R_h g_P - g_P) D_{i+1} D_i^2 u\, ds.$$

Let us fix some macro-edga Γ with end points a, b. Then, integration by parts yields

$$\int_{\Gamma_i} D_i(R_h g_P - g_P) D_{i+1}^2 u\, ds = -\int_{\Gamma_i}(R_h g_P - g_P) D_i D_{i+1}^2 u\, ds + (R_h g_P - g_P) D_{i+1}^2 u\Big|_a^b.$$

Hence, it remains to estimate quantities of the form

$$\int_\Gamma (R_h g_P - g_P)\psi\, ds, \qquad (R_h g_P - g_P)(b),$$

where Γ is a macro-edge, a a macro-vertice, and ψ stands for the trace of a fixed $C^{1+\alpha}$ function. Below in Lemma 6.9, we shall prove the error estimates

$$\int_\Gamma (R_h g_P - g_P)\psi\, ds = \mathcal{O}(h^2 L(h)), \tag{6.4.117}$$

$$(R_h g_P - g_P)(Q) = \mathcal{O}(h^2 L(h)), \tag{6.4.118}$$

for nodal points $Q \in \Omega$ having (uniformly) positive distance to P and the corner points of $\partial\Omega$. From these results, we infer that

$$\sum_j L_h^{(j)}(u, R_h g_P - g_P) = \mathcal{O}(h^2 L(h)). \tag{6.4.119}$$

Combining the estimates (6.4.116) and (6.4.119), we obtain

$$e_h(u, R_h g_P - g_P) = \mathcal{O}(h^2 L(h)).$$

This proves the desired error expansion

$$R_h u(P) = u(P) + h^2 \hat{e}(P) + h^2 e(P) + \mathcal{O}(h^4 L(h)), \qquad (6.4.120)$$

at nodal points $Q \in \Omega$ having (uniformly) positive distance to the macro-vertices and the corner points of $\partial\Omega$. Q.E.D.

Remark 6.7: The error expansion (6.4.103) has been proven only for nodal points bounded away from the vertices of the macro-triangles $\Omega^{(j)}$ and the corner points of $\partial\Omega$. Actually, it cannot hold true at those boundary vertices. However, for nodal points P in the neighborhood of macro-vertices A lying on the boundary (being even a corner point of $\partial\Omega$) an expansion with remainder term of reduced order can be proven:

$$R_h u(P) = u(P) + h^2 \hat{e}(P) + h^2 e(P) + \mathcal{O}(h^3 L(h)). \qquad (6.4.121)$$

For that, we need to modify the argument used in the proof of Theorem 6.8 only in estimating the terms $e_h(u, R_h g_P^h - g_P)$ and $L_h^{(j)}(u, R_h g_P^h - g_P)$, as the remainder term is always of the order $\mathcal{O}(h^4 L(h))$.

i) First, we notice that the error function e has globally H^2 regularity. Hence (exercise),

$$\|e - R_h e\|_{L^\infty} \le ch\|e\|_{H^2} = \mathcal{O}(h). \qquad (6.4.122)$$

ii) Next, we observe that for nodal points P as considered, integration by parts over a macro edge Γ_i with end points $A \in \partial\Omega$ and $B \in \bar{\Omega}$ yields

$$\int_{\Gamma_i} D_i(R_h g_P - g_P) D_{i+1}^2 u \, ds = -\int_{\Gamma_i} (R_h g_P - g_P) D_i D_{i+1}^2 u \, ds + (R_h g_P - g_P) D_{i+1}^2 u(B),$$

where the other end point b lies either also on the boundary $\partial\Omega$ (being possibly even a corner point), in which case $(R_h g_P^h - g_P) D_{i+1}^2 u(b) = 0$, or it is automatically bounded away from P and the corner points of $\partial\Omega$. Then, Lemma 6.9 below applies yielding

$$(R_h g_P - g_P) D_{i+1}^2 u(b) = \mathcal{O}(h^2 L(h)).$$

The remaining line integral, we estimate by the usual L^1-trace inequality combined with the L^1-Hölder inequality,

$$\Big| \int_{\Gamma_i} (R_h g_P - g_P) D_i D_{i+1}^2 u \, ds \Big| \le c\|u\|_{C^3} \|\nabla(R_h g_P - g_P)\|_{L^1}.$$

Thus, Lemma 6.9 implies that

$$\Big| \int_{\Gamma_i} (R_h g_P - g_P) D_i D_{i+1}^2 u \, ds \Big| = \mathcal{O}(hL(h)).$$

Combining the foregoing estimates, we obtain the asserted error expansion. We note that in the case of a globally three-directional mesh the expansion (6.4.121) holds uniformly at all nodal point $P \in \Omega$ with a remainder term of order $\mathcal{O}(h^3)$.

Remark 6.8: As a byproduct of Theorem 6.8, we obtain some information concerning the best possible order of convergence of *linear* finite elements. Let the macro-triangulation have only one interior vertex P and, using the notation from above, let the edges be numbered such that $P = P_3^{j)}$ for every macro-triangle $\Omega^{(j)}$ containing P. Then, from the proof of Theorem 6.8, we see that

$$P_h u(P) = u(P) + h^2 R_h g_P(P) \sum_j E_j(P) + \mathcal{O}(h^2),$$

where

$$E_j(P) = \frac{1}{2\alpha}\big((\lambda_2^4 D_2^2 u - \lambda_1^3 \lambda_3 n^1 \cdot n^3 D_1^2 u - (\lambda_3^4 D_3^2 u - \lambda_2^3 \lambda_1 n^2 \cdot n^1 D_2^2 u)\big)(P).$$

Consequently, if $\sum_j E_j(P) \neq 0$ the error in the nodal point P behaves asymptotically not better than

$$R_h u(P) = u(P) + \mathcal{O}(hL(h)),$$

since (exercise)

$$R_h g_P(P) \geq c|\ln(h)|.$$

This shows that, in general, the Ritz projection onto the space of *linear* finite elements does not allow for a better approximation order than $\mathcal{O}(h^2 L(h))$ even on smooth functions.

6.4.8 Error expansion on smoothly bounded domains

In the following, we treat the question of error expansion on a domain Ω with smooth boundary $\partial\Omega$. This makes the strong assumptions on the regularity of the solution more realistic but, on the other hand, complicates the finite element approximation. It is no longer possible to cover $\bar{\Omega}$ by a blockwise uniform triangulation $\mathbb{T}_h$. Instead, let $\Omega_h^0 \subset \Omega$ be an interior mesh region satisfying (see Fig. 6.6)

$$\mathrm{dist}(\Omega_h^0, \partial\Omega) = \mathcal{O}(h),$$

in which the triangulation is three-directional. In the remaining strip $B_h := \bar{\Omega} \setminus \Omega_h^0$ the triangulation is only assumed to be regular in the usual sense and to match the boundary $\partial\Omega$ of order $\mathcal{O}(h^2)$. The global mesh region is denoted by $\Omega_h := \cup\{T \in \mathbb{T}_h\} \subset \bar{\Omega}$. The corresponding finite element space is then defined by (assuming Ω to be convex)

$$V_h := \{v_h \in C(\bar{\Omega}) \,|\, v_h|_T \in P_1(T), T \in \mathbb{T}_h.\ v_h|_{\Omega\setminus\Omega_h} \equiv 0\} \subset H_0^1(\Omega).$$

The case that Ω is not convex, i. e., $\Omega_h \not\subset \bar{\Omega}$ can also be treated but requires more technical effort due to the resulting nonconformity of the discretization, $V_h \not\subset H_0^1(\Omega)$.

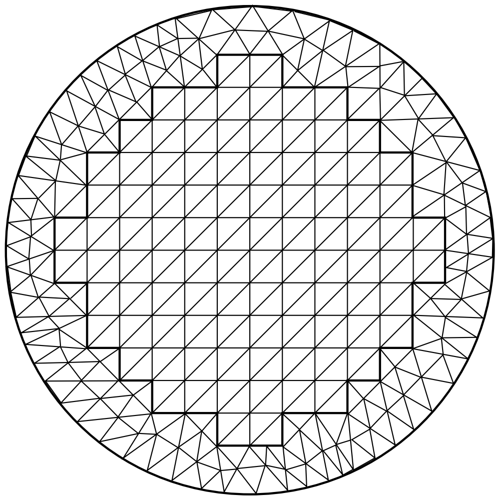

Figure 6.7: Interior uniform mesh with boundary approximation.

We want to derive error expansions at interior nodel points $P \in \Omega_h^0$. Here, a complication arises by the possible variation of the mesh structure in the boundary strip B_h in the course of mesh refinement. The practical consequences of this problem will be discussed below.

Theorem 6.9: *Let the solution be $u \in C^{4+\alpha}(\bar\Omega)$. Then, there holds an error expansion of the form*

$$R_h u(P) = u(P) + h^2 e_h(P) + \mathcal{O}(h^3 L(h)), \qquad (6.4.123)$$

for all interior nodel points $P \in \Omega' \subset\subset \Omega$, where the expansion coefficient $e_h(\cdot)$ depends on the structure of the mesh in the boundary strip B_h.

Proof: We only give a sketch of the proof and continue using the notation from the proof of Theorem 6.8. The starting point is again the following error representation at nodal points P:

$$(R_h u - u)(P) = (\nabla(u - I_h u), \nabla R_h g_P^h),$$

where, setting $\varphi_h := R_h g_P^h$,

$$(\nabla(u - I_h u), \nabla \varphi_h) = \sum_{T \in \mathbb{T}_h} \sum_{i=1}^{3} I_T^i,$$

and

$$\begin{aligned} I_T^i = {} & h^2 \beta_1 \frac{\lambda_i^4}{2\alpha} \int_{S_{i+2}} D_{i+2}\varphi_h D_i^2 u \, ds + h^2 \beta_1 \frac{\lambda_i^4 h_{i+1} \lambda_{i+2}}{4\alpha^2} \int_T \varphi_h D_{i+2} D_{i+1} D_i^2 u \, dx \\ & - h^2 \beta_1 \frac{\lambda_i^3 \lambda_{i+1}}{2\alpha} \int_{S_i} \varphi_h D_{i+1} D_i^2 u \, ds + h^2 \beta_1 \frac{\lambda_i^4}{2\alpha} \int_{S_{i+1}} \varphi_h D_{i+1} D_i^2 u \, ds \\ & - h^2 \beta_1 \frac{\lambda_i^3 \lambda_{i+2}}{2\alpha} n^i \cdot n^{i+2} \int_{S_i} D_i \varphi_h D_i^2 u \, ds + h^4 \lambda_i^4 \int_{S_i} \beta_2^i D_i^4 u \, \partial_{n^i} \varphi_h \, ds \,. \end{aligned}$$

i) As noted above all line integrals not belonging to the remainder (last term) cancel for edges interior to Ω_h^0. Therefore, we are left with the contribution

$$h^2 L_h(u, R_h g_P^h) = h^2 {\sum_j}' L_h^{(j)}(u, R_h g_P^h),$$

where $L_h^{(j)}(u, R_h g_P^h)$ is defined as in the preceding section, but now the index j refers to every triangle T separately. The sum ${\sum}_j'$ extends over all triangles containing the sides $S_i \subset B_h \cup \partial\Omega_h^0$.

ii) The sum over the area integrals results in an integral over Ω_h:

$$\begin{aligned} h^2 \hat{e}_h(u, \varphi_h) &= h^2 \beta_1 \sum_{T \in \mathbb{T}_h} \frac{\lambda_1\lambda_2\lambda_3}{4\alpha^2} \int_T \varphi_h \Big(D_1 D_2 D_3 \sum_{i=1}^{3} \lambda_i^3 D_i \Big) u \, dy \\ &= h^2 \int_{\Omega_h} \varphi_h D_h^{(4)}(u) \, dx \,. \end{aligned}$$

Since the characteristic quantities D_i, λ_i and α may be different for every $T \subset B_h$ the function $D_h^{(4)}(u)$ is only piecewise smooth near the boundary.

iii) The remainder term

$$h^4 \rho_h(u, R_h g_P^h) = h^4 \sum_{T \in \mathbb{T}_h} \sum_{i=1}^{3} \lambda_i^4 \int_{S_i} \beta_2^i D_i^4 u \, \partial_{n^i} \varphi_h \, ds$$

can be handled analogously to the preceding section yielding

$$h^4 \rho_h(u, R_h g_P^h) = h^4 \rho_h(u, R_h g_P^h - g_P^h) = \mathcal{O}(h^4 L(h)),$$

fior all nodal points $P \in \Omega_h$.

Using the foregoing results, we have the error representation

$$(R_h u - u)(P) = h^2 \{ \hat{e}_h(u, \varphi_h) + L_h(u, R_h g_P^h) \} + \mathcal{O}(h^4 L(h)). \tag{6.4.124}$$

Next, we define the expansion coefficient $e_h(P)$ again by replacing $R_h g_P^h$ by g_P:

$$e_h(P) := \hat{e}_h(u, g_P) + L_h(u, g_P).$$

This results in the error representation

$$(R_h u - u)(P) = h^2 e_h(P) + h^2 \{ \hat{e}_h(u, R_h g_P^h - g_P) + L_h(u, R_h g_P^h - g_P) \} + \mathcal{O}(h^4 L(h)).$$

The estimate

$$\hat{e}_h(u, R_h g_P^h - g_P) + L_h(u, R_h g_P^h - g_P) = \mathcal{O}(h L(h))$$

follows by the arguments from the proof of Theorem 6.8. The details are omitted. Q.E.D.

The coefficient $e_h(P)$ in the error expansion (6.4.123) depends on the structure of the mesh in the boundary strip B_h, which may change with the mesh size h. Hence its use as basis of extrapolation needs some care. To overcome this difficulty, we start from some basic mesh $\mathbb{T}_h$ and construct the refined mesh $\mathbb{T}_{h/2}$ by subdividing each triangle $T \in \mathbb{T}_h$ into four congruent subtriangles and then project the new nodal points on $\partial\Omega_h$ onto the curved boundary Ω. For this special refinement one step of extraplation yields

$$\tfrac{1}{3}\{4u_{h/2}(P) - u_h(P)\} = u(P) + \tfrac{1}{3}h^2\{e_{h/2}(P) - e_h(P)\} + \mathcal{O}(h^3L(h)). \qquad (6.4.125)$$

The difference of the two expansion coefficients vanishes up to the contributions from the boundary strips B_h and $B_{h/2}$, respectively. There are three different groups of term,s to estimates:

1. area integrals of the form

$$\sum_{T\in\mathbb{T}'_{h/2}} \frac{\lambda_1^4\lambda_2\lambda_3}{4\alpha^2}\int_T g_P D_1^4 D_2 D_3 u\,dx - \sum_{T\in\mathbb{T}'_h} \frac{\lambda_1^4\lambda_2\lambda_3}{4\alpha^2}\int_T g_P D_1^4 D_2 D_3 u\,dx,$$

where $\mathbb{T}'_h := \{T \in \mathbb{T}_h,\, T \subset B_h\}$,

2. line integrals of the form

$$\sum_{\Gamma_i\in S'_{h/2}} \frac{\lambda_{i+1}^4}{2\alpha}\int_{\Gamma_i} D_i g_P D_{i+1}^2 u\,ds - \sum_{\Gamma_i\in S_h} \frac{\lambda_{i+1}^4}{2\alpha}\int_{\Gamma_i} D_i g_P D_{i+1}^2 u\,ds$$

where $S_h := \{\Gamma_i,\, \Gamma_i \subset B_h \setminus \partial\Omega_h\}$ and $S'_{h/2} := \{\Gamma_i \in S_{h/2},\, \Gamma_i \subset \cup\{\Gamma \in S_h\}\}$,

3. line integrals of the form

$$\sum_{\Gamma_i\in S_{h/2}\setminus S'_{h/2}} \frac{\lambda_{i+1}^4}{2\alpha}\int_{\Gamma_i} D_i g_P D_{i+1}^2 u\,ds\,.$$

In the boundary strip $B_{h/2}$ two adjacent triangles do not necessarily form an accurate parallelogram. However, intagrals of the form

$$\int_{\Gamma_i} D_i g_P\{(t_{i+1}\cdot\nabla)^2 u - (t'_{i+1}\cdot\nabla)^2 u\}\,ds$$

over edges $\Gamma_i \in S_{h/2}\setminus S'_{h/2}$ are of order $\mathcal{O}(h^2)$, since $t_{i+1}+t'_{i+1} = \mathcal{O}(h)$ by the construction of $\mathbb{T}_{h/2}$. By arguments of this kind, we find that

$$e_{h/2}(P) - e_h(P) = \mathcal{O}(h).$$

In view of (6.4.125) his implies the extrapolation formula

$$\tfrac{1}{3}\{4u_{h/2}(P) - u_h(P)\} = u(P) + \mathcal{O}(h^3L(h)). \qquad (6.4.126)$$

Remark 6.9: For special configurations involving curved boundaries, such as curved ducts, nozzles or ring segments, almost three-directional triangulations (possibly consisting of triangles with curvedsides) can be constructed. On more general domains, one may use blockwise versions of such “curved” triangulations. On such meshes asymptotic error expansions may be derived similar to the ones stated above. This makes Richardson extrapolation applicable also on genarl domains with curved boundaries.

6.4.9 Further estimates for the Ritz projection of the Green's function

In Section 6.4.7, we have used some local error estimates for the Ritz projection of the “exact” Green's function g_P. These estimates are provided by the following lemma.

Lemma 6.9: *Let $\mathbb{T}_h$ be a quasi-uniform triangulation of $\bar\Omega$ and P be any nodal point.*
i) For any point $z\in\Omega$ having positive distance from P and the corner points of $\partial\Omega$ there holds

$$(R_h g_P - g_P)(z) = \mathcal{O}(h^2L(h)). \tag{6.4.127}$$

ii) Let $\Gamma\subset\bar\Omega$ be a straight line consisting of sides of triangles $T\in\mathbb{T}_h$ with end points a,b having both positive distance to P and the corner points of $\partial\Omega$, Then, for any function $\psi\in C^{1+\alpha}(\bar\Omega)$ there holds

$$\int_\Gamma (R_h g_P - g_P)\psi\,ds = \mathcal{O}(h^2L(h)). \tag{6.4.128}$$

If one of the end points a,b is close to P or to a corner point of $\partial\Omega$ the order of this estimate reduces to $\mathcal{O}(hL(h))$.

Proof: We only give a sketch of the proof.
i) Let $z\in T_z\in\mathbb{T}_h$ be a point with positive distance to P and to the corner points of $\partial\Omega$. With the regularized Green's function g_z^h considered in Section 6.4.6, we find using Galerkin orthogonality that

$$\begin{aligned}
(g_P - R_h g_P)(z) &= (g_P - \tilde I_h g_P)(z) + (\tilde I_h g_P - R_h g_P)(z)\\
&= (g_P - \tilde I_h g_P)(z) + (\nabla g_z^h, \nabla(I_h g_P - R_h g_P))\\
&= (g_P - \tilde I_h g_P)(z) + (\nabla g_z^h, \nabla(\tilde I_h g_P - g_P)) + (\nabla g_z^h, \nabla(g_P - R_h g_P))\\
&= (g_P - \tilde I_h g_P)(z) + (\delta_z^h, \tilde I_h g_P - g_P)_{T_z} + (\nabla(g_z^h - R_h g_z^h), \nabla(g_P - R_h g_P))\\
&= (g_P - \tilde I_h g_P)(z) + (\delta_z^h, \tilde I_h g_P - g_P)_{T_z} + (\nabla(g_z^h - R_h g_z^h), \nabla(g_P - \tilde I_h g_P)),
\end{aligned}$$

where $\tilde I_h g_P\in V_h$ denotes some nodal interpolant regularized at the singular point P. Observing that $\mathrm{dist}(T_z,P)>0$ (for sufficiently small h) and $g_P\in C^2$ in a neighborhood of T_z the interpolation errors on the right behave like $\mathcal{O}(h^2)$. Hence,

$$(g_P - R_h g_P)(z) = \mathcal{O}(h^2) + (\nabla(g_z^h - R_h g_z^h), \nabla(g_P - \tilde I_h g_P)). \tag{6.4.129}$$

For estimating the last term on the right, we need to separate the singularities of the Green's function at the points P and z. Let $B_P \subset B'_P \subset \bar{\Omega}$ and $B_z \subset B'_z \subset \bar{\Omega}$ be small circular mesh domains surrounding P and z, respectively, which satisfy

$$\operatorname{dist}(B'_P, B'_z) \geq \delta, \quad \operatorname{dist}(B_P, \partial B'_P) \geq \delta, \quad \operatorname{dist}(B_z, \partial B'_z) \geq \delta,$$

with some fixed constant $\delta > 0$. Set $\Sigma_h := \Omega \setminus \{B_P \cup B_z\}$ and let $\Sigma_h \subset \Sigma'_h \subset \bar{\Omega}\}$ with $\operatorname{dist}(\Sigma_h, \partial\Sigma'_h) \geq \delta$. (The reader may draw a picture of this situation.) Notice that

$$g_P \in H^2(\Omega \setminus B_P) \cap C^2(B_z).$$

With the above notation, we have

$$\begin{aligned}
(\nabla(g_z^h - R_h g_z^h), \nabla(g_P - \tilde{I}_h g_P)) &= (\nabla(g_z^h - R_h g_z^h), \nabla(g_P - \tilde{I}_h g_P))_{\Sigma_h} \\
&\quad + (\nabla(g_z^h - R_h g_z^h), \nabla(g_P - \tilde{I}_h g_P))_{B_P} \\
&\quad + (\nabla(g_z^h - R_h g_z^h), \nabla(g_P - \tilde{I}_h g_P))_{B_z} \\
&\leq \|\nabla(g_z^h - R_h g_z^h)\|_{L^2(\Sigma_h)} \|\nabla(g_P - \tilde{I}_h g_P)\|_{L^2(\Sigma_h)} \\
&\quad + \|\nabla(g_z^h - R_h g_z^h\|_{L^\infty(B_P)} \|\nabla(g_P - \tilde{I}_h g_P)\|_{L^1(B_P)} \\
&\quad + \|\nabla(g_z^h - R_h g_z^h\|_{L^1(B_z)} \|\nabla(g_P - \tilde{I}_h g_P)\|_{L^\infty(B_z)}.
\end{aligned}$$

We recall the following interpolation estimates:

$$\begin{aligned}
\|\nabla(g_P - \tilde{I}_h g_P)\|_{L^2(\Sigma_h)} &\leq \mathcal{O}(h), \\
\|\nabla(g_P - \tilde{I}_h g_P)\|_{L^1(B_P)} &\leq \mathcal{O}(hL(h)), \\
\|\nabla(g_P - \tilde{I}_h g_P)\|_{L^\infty(B_z)} &\leq \mathcal{O}(h),
\end{aligned}$$

where the proof of the second estimate requires some special care due to the singularity of g_p at P (exercise). For treating the remaining projection error terms, we state the following local L^2- and L^∞-error estimates and use known properties of the regularized Green's function g_z^h:

$$\begin{aligned}
\|\nabla(g_z^h - R_h g_z^h)\|_{L^2(\Sigma_h)} &\leq c\|\nabla(g_z^h - I_h g_z^h)\|_{L^2(\Sigma'_h)} + c\|g_z^h - R_h g_z^h\|_{L^2(\Omega)} \\
&\leq ch\|\nabla^2 g_z^h\|_{L^2(\Sigma'_h)} + ch^2\|\nabla^2 g_z^h\|_{L^2(\Omega)} = \mathcal{O}(h), \\
\|\nabla(g_z^h - R_h g_z^h)\|_{L^\infty(B_P)} &\leq c\|\nabla(g_z^h - I_h g_z^h)\|_{L^\infty(B'_P)} + c\|g_z^h - R_h g_z^h\|_{L^\infty(\Omega)} \\
&\leq ch\|\nabla^2 g_z^h\|_{L^\infty(B'_P)} + ch\|\nabla^2 g_z^h\|_{L^2(\Omega)} = \mathcal{O}(h), \\
\|\nabla(g_z^h - R_h g_z^h\|_{L^1(B_z)} &\leq \|\nabla(g_z^h - R_h g_z^h\|_{L^1(\Omega)} = \mathcal{O}(hL(h)).
\end{aligned}$$

The local error estimates can be derived by arguments similar to those used in proving the local stability estimate (6.4.80) of Lemma 6.5 (for full details see [113]). Collecting the foregoing estimates yield

$$(\nabla(g_z^h - R_h g_z^h), \nabla(g_P - \tilde{I}_h g_P)) = \mathcal{O}(h^2 L(h)),$$

and, consequently,

$$(g_P - R_h g_P)(z) = \mathcal{O}(h^2 L(h)),$$

what was to be shown.

ii) Next, we sketch the proof of the estimate (6.4.128). Let B_a and B_b mesh regions surrounding the endpoints of Γ such that $\mathrm{dist}(P, B_a \cup B_b) \geq \delta > 0$. Then, we can use the estimate (6.4.127) to obtain

$$\int_{\Gamma\cap\{B_a\cup B_b\}} (R_h g_p - g_P)\psi\, ds = \mathcal{O}(h^2 L(h)).$$

Hence, for the following, we can assume that $\psi = 0$ near a and b. Now, we use a duality argument. Let $v \in H_0^1(\Omega)$ be the solution of the auxiliar problem

$$(\nabla\varphi, \nabla v) = \int_\Gamma \varphi\psi\, ds \quad \forall\varphi \in H_0^1(\Omega). \tag{6.4.130}$$

Then, $v \in H^2(\Omega\setminus\Gamma) \cap C^2(\Omega_0 \setminus \Gamma)$ for any subdomain $\Omega_0 \subset \Omega$ having positive distance to the corner points of $\partial\Omega$. For the Ritz projection $R_h v \in V_h$ there holds

$$\|v - R_h v\| + h\|\nabla(v - R_h v)\| = \mathcal{O}(h^2),$$
$$\|\nabla(v - R_h v)\|_{L^\infty(\Omega_0)} = \mathcal{O}(h),$$

depending on the data ψ. One can justify setting $\varphi = R_h g_P - g_P$ in (6.4.114) to obtain

$$\int_\Gamma (R_h g_P - g_P)\psi\, ds = (\nabla(R_h g_P - g_P), \nabla v) = (\nabla(\tilde{I}_h g_P - g_P), \nabla(v - R_h v)).$$

Using here the above error estimates, we conclude the desired estimate (6.4.128). Q.E.D.

6.4.10 Numerical tests

For verifying the theoretical results of the preceding sections, the model problem (6.4.36) has been solved on several sevela triangular domains and on the unit square using various piecewise or locally uniform triangulations. Fig. 6.8 shows the coarsest triangulation, $\mathbb{T}_0$, corresponding to the mesh size $h = h_0$. The refined meshes $\mathbb{T}_k$, for $k \geq 1$, are constructed by successive decomposition of each triangle onto four congruent subtriangles of width $h_k = 2^{-k}h$. For configuration (VI) the triangulations $\mathbb{T}_k$ are kept uniform up to the boundary strip B_{h_k} of width h_k. The data f has been chosen such that the solution u is a polynomial. Although these tests cannot be considered to be exhaustive , they are certainly representative for the case of a smooth solution. The errors after one and two steps of extrapolation ($u_k := u_{h_k} := R_{h_k} u$),

$$u_k^1 = \tfrac{1}{3}\{4u_{h_{k+1}} - u_{h_k}\}, \quad u_k^2 = \tfrac{1}{3}\{4u^1_{h_{k+1}} - u^1_{h_k}\},$$

have been evaluated at interior nodal points P.

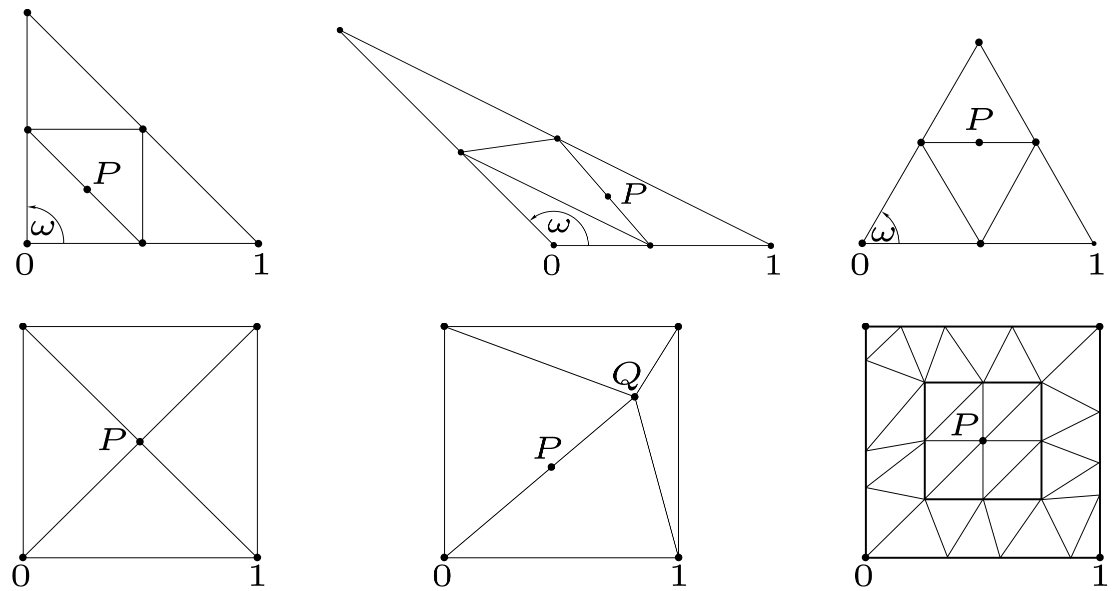

Figure 6.8: Configurations for the numericsl tests.

The following tables show the corresponding error quantities $\varepsilon_k := |u_k - u|$, $\varepsilon_k^1 := |u_k^1 - u|$ and $\varepsilon_k^2 := |u_k^2 - u|$, and the approximate orders of convergence m_k, m_k^1, m_k^2, which are calculated as usual by the formula $m_k = (\ln(\varepsilon_i) - \ln(\varepsilon_{k+1})/\ln(2)$. The theoretically presicted orders of convergence are listed as m_∞. We note that the error behavior shown in Tables 6.2 - 6.7 is representative for all nodal points belonging to the coarser meshes.

The tests I and II confirm the interior $\mathcal{O}(h^4)$ result in Theorem 6.7 and also indicate that an extended expansion of the form (6.4.65) may be valid. In test III, we see the $\mathcal{O}(h^4)$ super-convergence of the error $R_h u - u$ on equilateral meshes predicted in (6.4.88).

Table 6.2: Configuration I: $P = (\frac{1}{4}, \frac{1}{4})$.

k	ε_k	m_k	ε_k^1	m_k^1	ε_k^2	m_k^2
0	2.0 (-3)	1.962	1.7 (-5)	3.776	1.9 (-7)	5.532
1	5.1 (-4)	1.989	1.3 (-6)	3.930	4.2 (-9)	5.837
2	1.3 (-4)	1.997	8.3 (-8)	3.981	7.3 (-11)	5.952
3	3.2 (-5)	1.999	5.3 (-9)	3.995	1.2 (-12)	
4	8.0 (-6)	2.000	3.3 (-10)			
5	2.0 (-6)	$m_\infty = 2$		$m_\infty = 4$		$m_\infty = 6$

Table 6.3: Configuration II: $P = (\frac{1}{4}, \frac{1}{4})$.

k	ε_k	m_k	ε_k^1	m_k^1	ε_k^2	m_k^2
0	6.5 (-2)	1.895	1.6 (-3)	3.154	8.7 (-5)	4.076
1	1.7 (-2)	1.955	1,8 (-4)	3.493	5.1 (-6)	4.762
2	4.5 (-3)	1.984	1.6 (-5)	3.768	1.9 (-7)	5.142
3	1.1 (-3)	1.995	1.2 (-6)	3.906	5.4 (-9)	
4	2.8 (-4)	1.999	8.8 (-8)			
5	7.1 (-5)	$m_\infty = 2$		$m_\infty = 4$		$m_\infty = 5.3$

Table 6.4: Configuration III: $P = (\frac{1}{2}, \frac{\sqrt{3}}{4})$.

k	ε_k	m_k
0	8.3 (-4)	4.037
1	5.1 (-5)	4.008
2	3.2 (-6)	3.996
3	2.0 (-7)	3.971
4	1.3 (-8)	3.887
5	8.5 (-10)	$m_\infty = 4$

Test IV is intended to clarify the question of the optimal order of the pointwise convergence discussed in Remark 6.8. Although, the triangulation involves a considerable symmetry the numerical results clearly show the logarithmic error behavior $\mathcal{O}(h^2 L(h))$ for this particular configuration where $\sum_j E_j(P) = -4\Delta u(P) \neq 0$.

Table 6.5: Configuration IV: $P = (\frac{1}{2}, \frac{1}{2})$.

k	ε_k	m_k	ε_k / h_k^2	$\varepsilon_k / h_k^2 L(h_k)$
1	7.7 (-2)	0.913	0.309	0.445
2	4.1 (-2)	1.432	0.656	0.473
3	1.5 (-2)	1.610	0.972	0.465
4	5.0 (-3)	1.698	1.274	0.460
5	1.5 (-3)	1.752	1.571	0.453
6	4.6 (-4)		1.866	0.449

Test V confirms the result of Theorem 6.8 that the expansion 6.4.103) holds true with a remainder of order $\mathcal{O}(h^4 L(h))$ even at nodal points on the macro-edges where the uniformity of the mesh is broken.

Table 6.6: Configuration V: $P = (\frac{1}{2}, \frac{1}{2})$.

k	ε_k	m_k	ε_k^1	m_k^1	ε_k^2	m_k^2
1	4.5 (-2)	1.907	1.0 (-3)	3.679	1.7 (-5)	5.323
2	1.2 (-2)	1.972	7.8 (-5)	3.889	4.2 (-7)	6.065
3	3.1 (-3)	1.993	5.3 (-6)	3.975	6.2 (-9)	6.283
4	7.7 (-4)	1.998	3.4 (-7)	3.995	8.0 (-11)	
5	1.9 (-4)	2.000	2.1 (-6)			
6	4.8 (-5)	$m_\infty = 2$		$m_\infty = 4$		$m_\infty =?$

Finally, test VI stands as a model for the case of a smoothly bounded domain in so far as the uniformity of the mesh is perturbed in a boundary strip of width $\mathcal{O}(h)$. Further, it is guaranteed that the presence of the corners of $\partial\Omega$ does not effect the desired order of the error expansion since u is chosen in such a way that the expansion coefficient e is amooth up to the boundary. The results listed in Table 6.7 show that the local $\mathcal{O}(h^3 L(h))$ estimate for the remainder term in the expansion (6.4.123) of Theorem 6.9 is sharp. Further, a comparison of the orders m_k^1 for a sequence of points approaching the boundary indicates that the expansion cannot be extended up to the boundary.

Table 6.7: Configuration VI: $P = (\frac{1}{2}, \frac{1}{2})$.

k	ε_k	m_k	ε_k^1	m_k^1
0	4.0 (-2)	1.891	1.0 (-3)	3.917
1	1.1 (-2)	1.974	6.9 (-5)	2.649
2	3.1 (-3)	1.985	1.1 (-5)	2.949
3	7.7 (-4)	1.992	1.4 (-6)	2.666
4	1.9 (-4)	1.995	2.2 (-7)	$m_\infty = 3$

6.5 Defect correction in the FEM for elliptic PDE

For simplicity, we consider again the model problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{6.5.131}$$

on a convex polygonal domain $\Omega \subset \mathbb{R}^2$. Extensions to more complicated situations including variable coefficients and curved boundaries, fourth-order problems, second-order systems and even saddle-point problems are possible; see Blum [46], Lin & Xie [86], and Rannacher [104, 108].

Let $\mathbb{T}_h = \{T\}$ be structural regular triangulation of $\bar{\Omega}$ as introduced above. The family $\{\mathbb{T}_h\}_h$ is assumed to be quasi-uniform in the usual sense, i. e., to satisfy the uniform shape and size condition. The corresponding spaces of "linear" finite elements are $V_h \subset H_0^1(\Omega)$. The natural nodal interpolation operator is denoted by I_h while P_h and R_h denote the L^2 and the Ritz projection, respectively. Further, we introduce a "discrete" Laplacian operator $\Delta_h : V_h \to V_h$ through the relation

$$(-\Delta_h v_h, \varphi_h) = (\nabla v_h, \nabla \varphi_h) \quad \forall \varphi_h \in V_h.$$

With this notation the Ritz projection $R_h u \in V_h$ of the solution u of problem (6.5.131) is defined by

$$(\nabla R_h u, \nabla \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in V_h, \tag{6.5.132}$$

or in operator notation

$$-\Delta_h R_h u = P_h f. \tag{6.5.133}$$

For this scheme, we recall the well known local L^∞-error estimate

$$\|u - R_h u\|_{L^\infty(\Omega'')} \le ch^2 L(h)\big\{\|u\|_{H^{2,\infty}(\Omega')} + \|u\|_{H^2(\Omega)}\big\}, \tag{6.5.134}$$

on any fixed subdomains $\Omega'' \subset \Omega' \subset \Omega$, $\operatorname{dist}(\Omega'', \partial\Omega') \ge \delta > 0$, having positive distance from the set Σ of corner points of $\partial\Omega$, $\operatorname{dist}(\Omega', \Sigma) \ge \delta > 0$. This estimate holds on general quasi-uniform meshes. On certain more structured (blockwise three-directional) meshes it can be extended to an asymptotic error expansion on interior subregions $\Omega_0 \subset \Omega$ of the form

$$R_h u = I_h u + h^2 I_h e(u) + h^4 \rho_h(u), \tag{6.5.135}$$

where $e(u) \in H^2(\Omega) \cap C^{2+\alpha}(\Omega_0)$ is a fixed function depending on u and the remainder term satisfies

$$\|\rho_h(u)\|_{L^\infty(\Omega_0)} \le cL(h),$$

provided that the solution u is sufficiently smooth ($u \in C^{4+\alpha}(\bar{\Omega})$). On globally uniform meshes this expansion holds on all of Ω with a remainder term of order $\mathcal{O}(h^3)$. In many applications the accuracy of the second-order discretization (6.5.133) does not suffice. However, the use of higher-degree finite elements may be prohibited since the corresponding system matrix is expensive to generate and to invert. For the low-order scheme often espacially adapted efficient solvers (e. g., multigrid methods) are available. In this case the classical Richardson extrapolation or, alternatively, the defect correction approach provide ways for increasing the accuracy while retaining the solution economy of the base scheme. To outline this procedure, we start with its very simplest version.

6.5.1 Defect correction by higher-order interpolation

For a triangulation $\mathbb{T}_{2h}$ of cell-width $2h$ let a refined triangulation $\mathbb{T}_h$ be constructed by subdeviding each element into four congruent subtriangles. Then, in view of the expansion (6.5.135) the Richardson extrapolation constructs an improved approximation $u^*_{2h} \in V_{2h}$ by combining the two solutions $u_{2h} := R_{2h}u$ and $u_h := R_hu$:

$$u^*_{2h} = \tfrac{1}{3}\{4I_{2h}u_h - u_{2h}\} = I_{2h}u + \mathcal{O}(h^4L(h)). \tag{6.5.136}$$

This method requires the computation of the discrete solutions u_{2h} and u_h on the two meshes $\mathbb{T}_{2h}$ and $\mathbb{T}_h$, respectively, but yields an improved approximation only on the coarser mesh $\mathbb{T}_{2h}$. This shortcoming are avoided in the defect correction method using "higher-order interpolation", which works as follows:

On the coarser triangulation $\mathbb{T}_{2h}$, we introduce the space $V^{(2)}_{2h}$ of *quadratic* finite elements,

$$V^{(2)}_{2h} := \{v_{2h} \in H^1_0(\Omega),\ v_{2h}|_T \in P_2(T),\ T \in \mathbb{T}_{2h}\},$$

and denote by $I^{(2)}_{2h} : C(\bar\Omega) \to V^{(2)}_{2h}$ the corresponding nodal interpolation operator (with the vertices of $\mathbb{T}_h$ as nodal points). This approximation is of third order,

$$\|v - I^{(2)}_{2h}v\|_{L^\infty} \le ch^3\|v\|_{H^{3,\infty}}. \tag{6.5.137}$$

Now, having computed the approximate solution $R_hu \in V_h$ from (6.5.133), we lift it to the high-order space $V^{(2)}_{2h}$ by $I^{(2)}_{2h}R_hu \in V^{(2)}_{2h}$ and define the "defect" $d_h \in V_h$ by

$$d_h := -\Delta_hR_hI^{(2)}_{2h}R_hu - P_hf. \tag{6.5.138}$$

Then, we compute a "correction" $k_h \in V_h$ from the "correction equation"

$$-\Delta_hk_h = -d_h \tag{6.5.139}$$

and define the new improved approximation $u^*_h \in V_h$ by setting

$$u^*_h := R_hu + k_h. \tag{6.5.140}$$

Combining these steps, we see that u^*_h satisfies the equation

$$-\Delta_hu^*_h = -\Delta_hR_hu - \Delta_hk_h = P_hf - d_h = P_hf + \Delta_hR_hI^{(2)}_{2h}R_hu + P_hf \tag{6.5.141}$$

This equation reads in variational form as follows:

$$\begin{aligned}(\nabla u_h^*, \nabla\varphi_h) &= (P_h f, \varphi_h) - (\nabla R_h I_{2h}^{(2)} R_h u, \nabla\varphi_h) + (P_h f, \varphi_h)\\ &= 2(f, \varphi_h) - (\nabla I_{2h}^{(2)} R_h u, \nabla\varphi_h).\end{aligned}$$

Hence, the computation of $u_h^* \in V_h$ from u_h requires the evaluation of the defect form $(d_h, \varphi_h) = (\nabla I_{2h}^{(2)} R_h u, \nabla\varphi_h) - (f, \varphi_h)$ and one additional solution of the discrete problem on the mesh $\mathbb{T}_h$ (i. e., with the same system matrix corresponding to the operator Δ_h). In contrast to Richardson extrapolation the defect correction procedure may be iterated in order to improve its accuracy lifting effect (see, e. g., Frank et al. [66]).

Theorem 6.10: *Let the mesh $\mathbb{T}_h$ be three-directional and the solution satisfy $u \in C^{4+\alpha}(\bar\Omega)$. Then for the approximation $u_h^* \in V_h$ obtained by defect correction with higher-order (quadratic) interpolation there holds*

$$\|u_h^* - I_h u\|_{L^\infty} = \mathcal{O}(h^3 L(h)). \qquad (6.5.142)$$

Proof: The proof is based on the asymptotic error expansion

$$R_h u = I_h u + h^2 e(u) + \rho_h(u), \qquad \|\rho_h(u)\|_{L^\infty} = \mathcal{O}(h^3), \qquad (6.5.143)$$

which on a globally uniform mesh holds on all of Ω, i. e., the remainder term is a cellwise continuous function of order $\mathcal{O}(h^3)$. By construction and observing $R_h = R_h R_h$, $R_h I_h = I_h$, and $I_{2h}^{(2)} = I_{2h}^{(2)} I_h$, there holds

$$\begin{aligned}-\Delta_h(u_h^* - I_h u) &= P_h f + \Delta_h R_h I_{2h}^{(2)} R_h u + P_h f + \Delta_h I_h u\\ &= -\Delta_h R_h u + \Delta_h R_h I_{2h}^{(2)} R_h u - \Delta_h R_h u + \Delta_h R_h I_h u - [\Delta_h R_h I_{2h}^{(2)} u + \Delta_h R_h I_{2h}^{(2)} u]\\ &= -\Delta_h R_h R_h u + \Delta_h R_h I_h u + \Delta_h R_h I_{2h}^{(2)} R_h u - \Delta_h R_h I_{2h}^{(2)} I_h u + \Delta_h R_h I_{2h}^{(2)} u - \Delta_h R_h u\\ &= -\Delta_h R_h (R_h u - I_h u) + \Delta_h R_h I_{2h}^{(2)} (R_h u - I_h u) + \Delta_h R_h (I_{2h}^{(2)} u - u)\\ &= -\Delta_h R_h (I - I_{2h}^{(2)})(R_h u - I_h u) + \Delta_h R_h (I_{2h}^{(2)} u - u).\end{aligned}$$

Hence,

$$u_h^* - I_h u = R_h (I - I_{2h}^{(2)})(R_h u - I_h u) - R_h (I_{2h}^{(2)} u - u),$$

and, consequently,

$$\|u_h^* - I_h u\|_{L^\infty} \le \|R_h (I - I_{2h}^{(2)})(R_h u - I_h u)\|_{L^\infty} + \|R_h (I_{2h}^{(2)} u - u)\|_{L^\infty}. \qquad (6.5.144)$$

In virtue of the L^∞-stability estimate of the extended Ritz projection $R_h : C_0(\partial T_h) \to V_h$ proven in Section 6.4.4,

$$\|R_h v\|_{L^\infty} \le c L(h) \|v\|_{C(\partial\Omega)}, \quad v \in C_0(\partial\mathbb{T}_h),$$

it follows that (i)

$$\|R_h(I-I_{2h}^{(2)})(R_hu-I_hu)\|_{L^\infty} \le cL(h)\|(I-I_{2h}^{(2)})(R_hu-I_hu)\|_{C(\bar\Omega)},$$

and (ii)

$$\|R_h(I_{2h}^{(2)}u-u)\|_{L^\infty} \le cL(h)\|I_{2h}^{(2)}u-u\|_{C(\bar\Omega)}.$$

This yields

$$\|u_h^*-I_hu\|_{L^\infty} \le cL(h)\big\{\|(I-I_{2h}^{(2)})(R_hu-I_hu)\|_{C(\bar\Omega)} + \|(I_{2h}^{(2)}u-u)\|_{C(\bar\Omega)}\big\}. \qquad (6.5.145)$$

Using the error expansion (6.5.135) and observing $I_{2h}^{(2)}I_h = I_{2h}^{(2)}$, we obtain

$$\begin{aligned}\|(I-I_{2h}^{(2)})(R_hu-I_hu)\|_{C(\bar\Omega)} &= \|(I-I_{2h}^{(2)})(h^2I_he(u)+\rho_h(u))\|_{C(\bar\Omega)}\\ &\le h^2\|(I_he-I_{2h}^{(2)})e(u)\|_{C(\bar\Omega)} + \|(I-I_{2h}^{(2)})\rho_h(u))\|_{C(\bar\Omega)}.\end{aligned}$$

In view of the interpolation estimate

$$\|I_he(u)-e(u)\|_{C(\bar\Omega)} + \|I_{2h}^{(2)}e(u)-e(u)\|_{C(\bar\Omega)} \le ch\|e(u)\|_{H^2} \le c(u)h,$$

we have

$$\|(I_he-I_{2h}^{(2)})e(u)\|_{C(\bar\Omega)} \le c(u)h.$$

Further, the bound $\|I_{2h}^{(2)}\rho_h(u)\|_{C(\bar\Omega)} \le c\|\rho_h(u)\|_{C(\bar\Omega)}$ implies that

$$\|(I-I_{2h}^{(2)})\rho_h(u))\|_{C(\bar\Omega)} \le c\|\rho_h(u))\|_{C(\bar\Omega)} \le c(u)h^3.$$

Combining the foregoing estimates, we obtain (i)

$$\|(I-I_{2h}^{(2)})(R_hu-I_hu)\|_{C(\bar\Omega)} \le c(u)h. \qquad (6.5.146)$$

By the interpolation estimate (6.5.137), we have (ii)

$$\|I_{2h}^{(2)}u-u\|_{C(\bar\Omega)} \le c(u)h^3. \qquad (6.5.147)$$

Using the last two estimates in (6.5.144), we eventually obtain

$$\|u_h^*-I_hu\|_{L^\infty} \le c(u)h^3L(h).$$

which completes the proof. Q.E.D.

Remark 6.10: The limitation to the order $\mathcal{O}(h^3L(h))$ in the error estimate (6.5.142) is due to the only third-order of approximation by quadratic finite elements (and also the lacking regularity of the expansion coefficient $e(u)$ near the corner points of $\partial\Omega$). By a refined analysis exploiting the super-approximation properties of quadratic elements on uniform meshes $\mathbb{T}_h$ the result can be improved to order $\mathcal{O}(h^4L(h))$ on interior subdomains $\Omega_0 \subset \Omega$. This is confirmed by the results of the numerical test presented below (see Table 6.8). However, a more robust approach for achieving this high-order accuracy is to use cubic finite elements in the defect computation (see Fig. 6.9). Then, the same line of argument as used above yields an error estimate of order $\mathcal{O}(h^4L(h)$ in interior subdomains $\Omega_0 \subset \Omega$ (exercise).

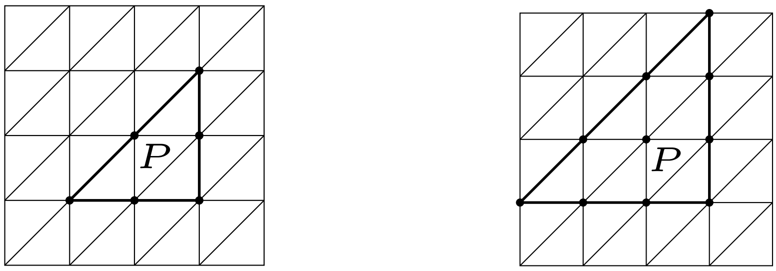

Figure 6.9: Nodal points of piecewise quadratic (left) and cubic (right) Lagrangian interpolation.

Remark 6.11: If the mesh $\mathbb{T}_h$ is only blockwise uniform the positive effect of defect correction breaks down in the neighborhood of interior macro-vertices. This can be improved by iterating the defect correction process one more step (see Table 6.9). Iterated defect correction may also be used in connection with defect computation with high-order (e. g., quintic) finite elements. However, this requires an even higher degree of regularity of the solution u, which limits the applicability of this approach.

The defect correction method described above has several advantages over the direct use of higher-order finite elements:

– The system matrices have a smaller number of nonzero entries compared, e. g., with the cubic fine element.

– The low-order base scheme can be very efficiently solved by the multigrid method.

– The availability of two approximations $u_h = R_h u$ and u_h^* allows an estimation of the discretization error: $u_h^* - I_h u = \frac{1}{3}(u_h - u_h^*) + \mathcal{O}(h^4L(h))$.

– The correction operator L_h^* is not required to be stable for $h \to 0$ (only of high-order consistent), which is important in the discretization of so-called "mixed" problems (such as mixed formulations of the plate problem and the "incompressible" Navier-Stokes equations).

Of course, a major disadvantage of the defect correction method is its lower flexibility, since its success depends on the uniformity of the mesh used, which is a problem in the case of complicated domains and curved boundaries.

Remark 6.12: In order to deal with problems posed on more complicated domains, with partially curved boundary, one may use "parametric" elements. First, a macro-decomposition $\mathbb{T}_H$ of $\bar{\Omega}$ of width $H \approx 1$ is chosen such that each macro-cell can be smoothly mapped onto a fixed reference cell $\hat{T}$. The macro-cells may be triangles or quadrilaterals. Along curved parts of the boundary $\partial\Omega$ the macro-cells are supposed to be likewise curved matching the boundary. Then, any uniform decomposition of $\hat{T}$ induces smoothly distorded uniform decompositions of each of the macro-cells and thereby a blockwise uniform decomposition of $\bar{\Omega}$. On such decompositions (consisting of triangles and/or quadrilaterals), one has asymptotic error expansions for *linear/bilinear* finite elements of the form

$$R_h u = I_h u + h^2 I_h e(u) + h^4 \rho_h(u),$$

on subdomains $\Omega_0 \subset \Omega$ bounded away from the macro-vertices. The error analysis developed above for the defect correction then yields the following optimal-order error estimate for cubic finite elements:

$$\|u_h^* - I_h u\|_{L^\infty(\Omega_0)} = \mathcal{O}(h^4 L(h)).$$

This is confirmed by numerical tests. Further, the reduced accuracy in the neighborhood of the macro-vertices can be essentially improved by performing a small number $(2-3)$ of additional local defect correction steps (see Table 6.10).

6.5.2 Numerical tests

The model problem (6.5.131) is considered a) on the unit square $\Omega = (0,1)^2$ with a uniform and a blockwise uniform triangulation and b) on the 1st quadrant of the unit circle on a (smoothly distorded) blockwise uniform quadrilateral mesh (see Fig. 6.10), in both cases with a smooth solution.

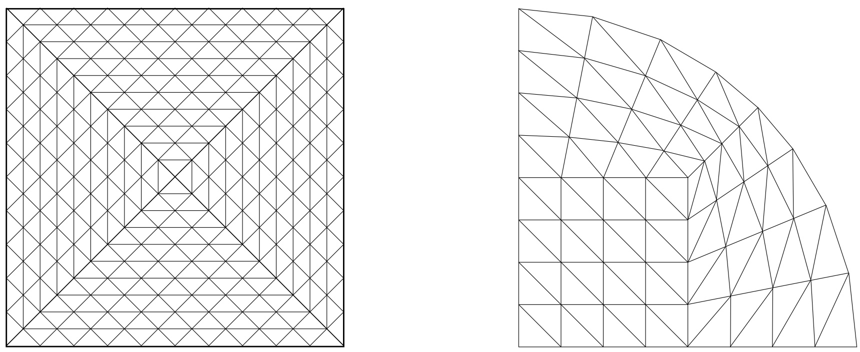

Figure 6.10: Blockwise uniform triangulation of the unit square and blockwise uniform quadrilateral decomposition of the 1st quadrant of the unit circle.

Table 6.8 shows the L^∞-errors and the corresponding orders m:

a) for the base scheme (6.5.132) and after one step of defect correction on the uniform mesh with quadratic elements,

b) after one step of defect correction on the blockwise uniform mesh, measured over all of Ω and then also measured over a subdomain $\Omega_0 \subset \Omega$ having positive distance $\delta = \frac{1}{8}$ from the macro-vertex.

Table 6.8: Errors and orders for defect correction with quadratic elements on uniform (a) and blockwise uniform (b) meshes. .

h	a) $\|e_h\|_\infty$	m	a) $\|e_h^*\|_\infty$	m	b) $\|e_h\|_\infty$	m	b) $\|e_h^*\|_\infty$	m
2^{-3}	7.41 (-3)		1.69 (-3)		2.90 (-3)		1.01 (-3)	
2^{-4}	1.95 (-3)	1.98	1.10 (-4)	3.95	6.10 (-4)	2.25	1.30 (-4)	2.95
2^{-5}	4.91 (-4)	1.99	6.93 (-6)	3.99	1.44 (-4)	2.08	1.66 (-5)	2.98

The following Table 6.9 contains some results obtained:

a) by directly using the cubic Hermite element in the base scheme,
b) by extrapolation applied to the *linear* base scheme,
c) by defect correction with the cubic Hermite element on a uniform mesh,
d) by the same method as under (c) but on a blockwise uniform mesh.

Table 6.9: Errors and orders for (a) direct computation with the cubic Hermite element, (b) extrapolation on a uniform mesh, (c) defect correction with the cubic Hermite element on a uniform mesh, and (d) the same method on a blockwise uniform mesh.

h	a) $\|e_h\|_\infty$	m	b) $\|e_h^*\|_\infty$	m	c) $\|e_h^*\|_\infty$	m	d) $\|e_h^*\|_{\infty;\Omega_0}$	m
2^{-3}	2.45 (-4)		1.55 (-4)		5.68 (-3)		5.52 (-5)	
2^{-4}	3.22 (-5)	2.93	1.09 (-5)	3.28	5.66 (-4)	3.32	4.30 (-6)	3.68
2^{-5}	2.53 (-6)	3,76	7.05 (-6)	3.95	4.14 (-5)	3.77	2.84 (-7)	3.92

Finally, Table 6.10 shows some results obtained by iterated defect correction with bi-cubic interpolation on blockwise uniform quadrilateral meshes of the 1st quadrant of the unit circle (see Fig. 6.10). In order to cope with the order reduction from $\mathcal{O}(h^4L(h))$ to $\mathcal{O}(h^2L(h))$ at the interior macro-vertex, one has to iterate the defect correction in order to get a uniformly small error. In virtue of the contraction property of the iteration operator a satisfactory quality is normally achieved after only $2-3$ steps. Since on the larger subdomain Ω_0 the 4th order accuracy is already guaranteed for the first iterate $u_h^{(1)} := u_h^*$ the further correction steps need to be performed only on a small neighborhood of the interior macro vertex.

Table 6.10: Global L^∞ errors for 5-times iterated defect correction with bi-cubic interpolation on blockwise uniform quadrilateral meshes of the 1st quadrant of the unit circle.

h	$\|e_h\|_\infty$	$\|e_h^{(1)}\|_\infty$	$\|e_h^{(2)}\|_\infty$	$\|e_h\|_\infty^{(3)}$	$\|e_h^{(4)}\|_\infty$	$\|e_h^{(5)}\|_\infty$
2^{-3}	1.25 (-2)	2.99 (-3)	7.10 (-4)	1.44 (-4)	1.65 (-4)	6.35 (-5)
2^{-4}	4.98 (-3)	7.19 (-3)	1.91 (-4)	5.88 (-5)	2.14 (-5)	1.04 (-5)
2^{-5}	1.53 (-3)	1.76 (-4)	4.65 (-5)	1.41 (-5)	4.86 (-6)	2.17 (-6)

Remark 6.13: The results discussed so far concern only the linear C^0-element in two dimensions. However, there seems to be no problem in extending them to the (isoparametric) bi-linear C^0-element on quadrilateral meshes, and to the corresponding nonconforming counterparts. For the latter, optimal L^1-error estimates for the regularized Green's function are available in the literature (see Goldstein [69] and Duran et al. [63]). Error expansions for nonconforming elements could be of particular value in approximating problems, which involve the incompressibility constraint $\nabla\cdot v=0$. The consideration of elements of higher than first order for Richardson extrapolation is of minor interest, because of their higher numerical complexity and the even stronger requirements on the solution's regularity; super-approximation results and error expansions for the quadratic triangular element were derived by Lin/Xie [85]. The methods described above for proving error expansions largely extend also to three dimensions, though a rigorous analysis has not been given yet in the literature.

6.6 Error expansions for other kinds of problems

6.6.1 Extrapolation in solving eigenvalue problems

We consider the approximation of the eigenvalue problem corresponding to the model problem (6.4.36),

$$-\Delta w=\lambda w \quad \text{in } \Omega, \qquad w=0 \ \text{ on } \partial\Omega, \tag{6.6.148}$$

where $w\in H_0^1(\Omega)$, $w\neq 0$, and $\lambda\in\mathbb{R}$. Since the Laplacian $-\Delta$ is a symmeric and positive definite operator in $L^2(\Omega)$ with compact inverse there are infinitely many positive eigenvalues λ without finite accumulation point and all eigenspaces $E_\lambda\subset H_0^1(\Omega)\cap H^2(\Omega)$ are finite dimensional. Let the eigenvalues be numbered like $0<\lambda=\lambda^{(1)}\le\lambda^{(2)}\le\dots$, and the associated eigenfunctions be normalized, $\|w\|=1$. The variational formulation of the eigenvalue problem reads

$$(\nabla w,\nabla\varphi)=\lambda(w,\varphi) \quad \forall\varphi\in H_0^1(\Omega). \tag{6.6.149}$$

The finite element approximation of this problem uses again subspaces $V_h\subset H_0^1(\Omega)$ of picewise *linear* functions on regular meshes $\mathbb{T}_h$ on $\bar\Omega$:

$$(\nabla w_h, \nabla\varphi_h) = \lambda_h(w_h, \varphi_h) \quad \forall\varphi_h \in V_h. \tag{6.6.150}$$

These discrete eigenvalue problems also possess only real positive eigenvalues $0 < \lambda_h := \lambda_h^{(1)} \leq \dots \lambda_h^{(N)}$ ($N = \dim V_h$), with corresponding eigenfunctions $w_h \in V_h$ normalized by $\|w_h\| = 1$. For these there hold the low-order error estimates

$$|\lambda - \lambda_h| + \|w - w_h\| = \mathcal{O}(h^2). \tag{6.6.151}$$

These estimates hold on convex polygonal domains or on domains with smooth boundary.

Theorem 6.11: *Let Ω have a smooth boundary and let the mesh $\mathbb{T}_h$ be blockwise (parametrical) uniform as described above. Then, for the smallest eigenvalue λ_h of problem (6.6.150) and the smallest eigenvalue λ of problem (6.6.149), there holds an asymptotic error expansion of the form*

$$\lambda_h = \lambda + \lambda\gamma(w)h^2 + \mathcal{O}(h^4 L(h)). \tag{6.6.152}$$

In the case of a convex polygonal domain Ω the error expansion holds true with a remainder of reduced order at least $\mathcal{O}(h^3)$.

Proof: We only give a sketch of the proof; for details see Lin/Xie [85] and Blum [46]. First, we note that

$$\lambda = \lambda\|w\|^2 = \|\nabla w\|^2, \qquad \lambda_h = \lambda_h\|w_h\|^2 = \|\nabla w_h\|^2,$$

and further

$$\lambda(w_h, w) = (\nabla w_h, \nabla w) = (\nabla w_h, \nabla R_h w) = \lambda_h(w_h, R_h w).$$

Using these identities, we can establish the following equation:

$$\lambda_h - \lambda = \|\nabla(w - R_h w)\|^2 + \lambda(w - w_h, R_h w - w) + (\lambda - \lambda_h)(w_h, R_h w - w_h). \tag{6.6.153}$$

Indeed, there holds

$$\begin{aligned}
\|\nabla(w - R_h w)\|^2 &+ \lambda(w - w_h, R_h w - w) + (\lambda - \lambda_h)(w_h, R_h w - w_h) \\
&= \|\nabla w\|^2 - 2(\nabla w, \nabla R_h w) + \|\nabla R_h w\|^2 \\
&\quad + \lambda\{(w, R_h w) - \|w\|^2 - (w_h, R_h w) + (w_h, w)\} \\
&\quad\quad + (\lambda - \lambda_h)\{(w_h, R_h w) - \|w_h\|^2\} \\
&= \|\nabla w\|^2 - 2(\nabla w, \nabla R_h w) + \|\nabla R_h w\|^2 \\
&\quad + \lambda\{(w, R_h w) - \|w\|^2 - (w_h, R_h w) + (w_h, w)\} \\
&\quad\quad + \lambda\{(w_h, R_h w) - \|w_h\|^2\} - \lambda_h\{(w_h, R_h w) - \|w_h\|^2\} \\
&= \lambda - 2\lambda(w, R_h w) + \lambda(w, R_h w) \\
&\quad + \lambda(w, R_h w) - \lambda - \lambda(w_h, R_h w) + \lambda(w_h, w) \\
&\quad\quad + \lambda(w_h, R_h w) - \lambda - \lambda_h(w_h, R_h w) + \lambda_h = -\lambda + \lambda_h.
\end{aligned}$$

The last two terms on the right in (6.6.153) are of the desired 4th order and can be absorbed into the remainder term (on general quasi-uniform meshes):

$$\begin{aligned}\lambda(w-w_h, R_hw-w) &\le \lambda\|w-w_h\|\|R_hw-w\| \\ &\le c\lambda h^4\|w\|_{H^2}^2 = \mathcal{O}(h^4), \\ (\lambda-\lambda_h)(w_h, R_hw-w_h) &\le |\lambda-\lambda_h|\|w_h\|\|R_hw-w_h\| \\ &\le |\lambda-\lambda_h|\{\|R_hw-w\|+\|w-w_h\|\} = \mathcal{O}(h^4).\end{aligned}$$

The first term can be rewritten as

$$\begin{aligned}\|\nabla(w-R_hw)\|^2 &= (\nabla(w-R_hw), \nabla(w-R_hw)) \\ &= (\nabla w, \nabla(w-R_hw)) = \lambda(w, w-R_hw) \\ &= \lambda(w, w-I_hw) + \lambda(w, I_hw-R_hw).\end{aligned}$$

For the term

$$(w, w-I_hw) = \sum_{T\in\mathbb{T}_h}\int_T w(w-I_hw)\,dx,$$

we can use an asymptotic error expansion for the 2-dimensional trapezoidal rule separately on each triangle observing that $w(w-I_hw)$ vanishes in all vertices. Since the mesh $\mathbb{T}_h$ is blockwise uniform this results in an expansion of the form

$$(w, w-I_hw) = ah^2 + \mathcal{O}(h^r),$$

for $w \in H^{r,1}(\Omega)$, $r \in \{3,4\}$, with a blockwise continuous function a. Notice that $w \in H^{4,1}(\Omega)$ on a smoothly bounded domain and $w \in H^{3,1}(\Omega)$ on a convex polygonal domain. In the remaining term (w, I_hw-R_hw), we use the error expansion

$$R_hw = I_hw + h^2e + h^2(R_he-e) + \mathcal{O}(h^4L(h)),$$

which holds on blockwise (parametrically) uniform meshes under the smoothness assumption $w \in C^{4+\alpha}(\bar{\Omega})$, which is justified on a smoothly bounded domain. On convex polygonal domains the argument has to be modified in order to avoid this strong regularity requirement. Here, the function e is blockwise in C^2 but at interior macro-vertices P contains the Green's function g_P. Therefore, in estimating the effect of the difference R_he-e the critical term is

$$\begin{aligned}h^2\lambda(w, R_hg_P-g_P) &= h^2(\nabla w, \nabla(R_hg_P-g_P)) \\ &= h^2(\nabla(w-R_hw), \nabla(R_hg_P-g_P)) \\ &= -h^2(\nabla(w-R_hw), \nabla g_P) = h^2(R_hw-w)(P).\end{aligned}$$

Since P is an interior nodal point, we have $(R_hw-w)(P) = \mathcal{O}(h^2L(h))$. Q.E.D.

We note that error expansions for eigenfunctions are also possible. Further, higher-order eigenvalues $\lambda^{(k)}$ and their approximations $\lambda_h^{(k)}$ error expansions shold with constants proportional to the size of $\lambda^{(k)}$. The expansion (6.6.152) can be taken as basis for extrapolation in eigenvalue approximation:

$$\lambda_h^* := \tfrac{1}{3}\{4\lambda_h - \lambda_{2h}\} = \lambda + \mathcal{O}(h^4L(h)). \tag{6.6.154}$$

6.6.2 Linear elliptic systems

As model situation of an elliptic system one may consider the Lamé-Navier system of plain linear elasticity

$$(\lambda+\sigma)\nabla(\nabla\cdot\bar{u})+\sigma\Delta\bar{u}=\bar{f}\quad\text{in }\Omega,\qquad \bar{u}=0\ \text{ on }\partial\Omega, \tag{6.6.155}$$

where, again, Ω is assumed to be a convex polygonal domain in $\mathbb{R}^2$. Here, $\bar{u}=(u_1,u_2)$ is the unknown displacement, and $\bar{f}$ is the acting body force assumed to be sufficiently regular. This system may be written in variational form in the Hilbert space $V:=H_0^1(\Omega)^2$ analogously as the model problem (6.4.36):

$$a(\bar{u},\bar{\varphi})=(\bar{f},\bar{\varphi})\quad\forall\bar{\varphi}\in V. \tag{6.6.156}$$

with "energy form"

$$a(\bar{u},\bar{\varphi}):=\int_\Omega\{\lambda\,\mathrm{tr}(\varepsilon[\bar{u}])\mathrm{tr}(\varepsilon[\bar{\varphi}])+2\sigma\,\varepsilon[\bar{u}]:\varepsilon[\bar{\varphi}]\}\,dx,$$

and the strain tensor $\varepsilon[\bar{u}]:=\frac{1}{2}(\nabla\bar{u}+\nabla\bar{u}^T)$. Then, the derivation of asymptotic error expansions for the corresponding Ritz projection $R_h\bar{u}$ follows the same line of argument as described above for the scalar model problem, now using expansions of the kind (6.4.64) also for $\mu\neq\nu$ (see [103] for more details).

Mixed formulation of the bi-harmonic problem

Another class of systems to which the above approach applies arises for instance in plate bending theory when the 4th-order biharmonic problem (simplified model of a thin elastic "clamped" plate under small loading)

$$\Delta^2u=f\quad\text{on }\Omega,\qquad u=\partial_nu=0\ \text{ on }\partial\Omega, \tag{6.6.157}$$

is written as a second-order system for $u\in H_0^1(\Omega)$ and the auxiliary variable $v:=-\Delta u$:

$$-\Delta u=v,\qquad -\Delta v=f.$$

On the convex polygonal domain the solution of the biharmonic problem satisfies $u\in H^3(\Omega)$ so that the regularity requirement $v\in H^1(\Omega)$ is justified. Then, the corresponding variational formulation can be written as

$$(\nabla\psi,\nabla u)-(\psi,v)+(\nabla u,\nabla\varphi)=(f,\varphi)\quad\forall\{\varphi,\psi\}\in H_0^1(\Omega)\times H^1(\Omega). \tag{6.6.158}$$

In this so-called "mixed" formulation the first boundary condition $u|_{\partial\Omega}=0$ is enforced by seeking $u\in H_0^1(\Omega)$, while the second boundary condition $\partial_nu|_{\partial\Omega}=0$ is built in as a "natural" boundary condition. This mixed formulation is well-posed. It may be approximated by *linear* finite elements (known as the so-called "Ciarlet-Raviart scheme", Ciarlet [31]),

$$(\nabla\psi_h, \nabla u_h) - (\psi_h, v_h) + (\nabla u_h, \nabla\varphi_h) = (f, \varphi_h) \quad \forall\{\varphi_h, \psi_h\} \in V_{h,0} \times V_h, \qquad (6.6.159)$$

with subspaces $V_h \subset H^1(\Omega)$ and $V_{h,0} \subset H_0^1(\Omega)$. For this discretization, we have the L^∞-error estimate (see Scholz [126])

$$\|u - R_h\|_\infty + \|v - R_v\|_{\infty;\Omega'} = \mathcal{O}(h^2 L(h)), \qquad (6.6.160)$$

where again $\Omega' \subset \Omega$ is a subdomain having positive distance to the corner points of $\partial\Omega$. The proof of corresponsing error expansions for the Ritz projections R_u and $R_h v$ (i. e., the solutions of the discrete problem (6.6.159) starts from the error identity

$$(\nabla\psi_h, \nabla(u - R_u)) - (\psi_h, u - R_h u) + (\nabla(v - R_h v), \nabla\varphi_h) = 0, \qquad (6.6.161)$$

for $\{\varphi_h, \psi_h\} \in V_{h,0} \times V_h$. In following the argument developed above for the scalar model problem (6.4.36), one now needs an expansion for the L^2 inner product $(\psi_h, v - R_h v)$. The proof of the analogue of the stability estimate (6.4.69) for $R_h u$ and $R_h v$ is again based on sharp L^1-error estimates for the discrete Green's function corresponding to the mixed formulation (6.6.159). Then, one obtains the expansion analogous to (6.4.66),

$$R_h u(P) = u(P) + h^2 e(P) + \mathcal{O}(h^{3.5}), \qquad (6.6.162)$$

at interior nodal points $P \in \Omega_0 \subset \Omega$, provided the solution u is smooth up to the boundary (see [103, 104]). Numerical tests indicate that that a corresponding expansion also holds for $R_h v$.

Remark 6.14: Similar results as discussed above for the Ciarlet-Raviart scheme also hold for the so-called Herrmann-Miyoshi scheme (see [104]), which is based on a mixed formulation of problem (6.6.157) with the tensor of bending moments, $M \sim \nabla^2 u$, as the secondary unknown. Error expansions for a mixed formulation of the Poisson equation are proven by Wang [134]. In view of the available L^∞ error estimates for the Stokes system in fluid mechanics error expansions are also possible for this kind of elliptic system.

6.6.3 Extrapolation in the presence of corner singularities

Standard finite element schemes usually suffer from a global loss of accuracy when used in the presence of reentrant corners (corner points of $\partial\Omega$ with interior angle $\omega > \pi$) or changing boundary conditions (e. g., sudden switch from Dirichlet to Neumann conditions). For suppressing this so-called "pollution effect" several methods have been considered in the literature, either using local mesh refinement or exokicitely incorporating the local singularities into the scheme. Here, an alternative approach is discussed in which the dominant error components are eliminated by Richardson extrapolation. This is theoretically justified through the derivation of asymptotic error expansions in terms of certain fractional powers of the mesh size depending on the type of occuring corner singularity.

We consider again the model problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega, \tag{6.6.163}$$

on a polygonal domain $\Omega \subset \mathbb{R}^2$ possessing a reentrant corner with interior angle $\omega > \pi$. If (6.6.163) is approximated by *linear* finite elements the observed order of pointwise convergence at interior nodes reduces to $\mathcal{O}(h)$ in the worst case of a "slit domain" with $\omega = 2\pi$. This contrasts to $\mathcal{O}(h^2)$ what is the order of local consistency. Here, "accuracy reduction" means that for solving the model problem with an error of 1% requires about $13\,000$ nodes on the slit domain Ω_2 shown in Fig. 6.11 compared to only 200 nodes on the "regular" domain Ω_1.

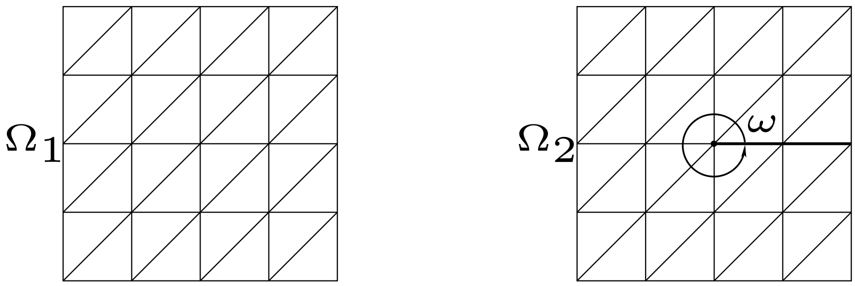

Figure 6.11: Unit square and slit domain ($\omega = 2\pi$).

This "pollution effect" cannot be avoided by use of higher-order elements as it depends on the approximability of the solution of a certain "dual" problem, which inevitably is also affected by the boundary singularity. In general, the presence of a reentrant corner z of the domain (assuming for simplicity that there is only one) with an interior angle $\omega \in (\pi, 2\pi]$ causes the solution of (6.6.163) to behave in the neighborhood of the corner point like $u(x) \sim r^\alpha$ as $r := \text{dist}(x, z) \to 0$, where $\alpha = \pi/\omega < 1$. In this case the best possible global error behavior is

$$\|\nabla(u - R_h u)\| = \min_{\varphi \in V_h} \|\nabla(u - \varphi_h)\| = \mathcal{O}(h^\alpha), \tag{6.6.164}$$

$$\|u - R_h u\|_\infty = \mathcal{O}(h^\alpha). \tag{6.6.165}$$

In interior subdomains $\Omega' \subset \Omega$ having positive distance to the corner point z the order of convergence is improved but only to $\mathcal{O}(h^{2\alpha})$. The "rule of thumb" is that the best order of pointwise convergence in the interior is at most twice the order of convergence in the global "energy norm":

$$\|u - R_h u\|_{\infty;\Omega'} \sim \|\nabla(u - R_h u)\|^2 = \mathcal{O}(h^{2\alpha}). \tag{6.6.166}$$

Several methods have been devised in the literature for coping with this problem. A proper local mesh refinement near the critical corner points guarantees that the "dual problem" can be approximated with optimal order, and thereby may recover the full global accuracy of the scheme (see Babuska [40], Schatz/Wahlbin [125], and E/Huang/Han [64]. This procedure requires the a priori knowledge of only the asymptotic growth of the singularities.

If the form of the singular parts of the solution is explicitly known one can incorporate them into the discretization by augmenting the trial space by these "singular" functions and/or by adding the corresponding "dual" singular functions as described in Remark 6.15, below, to the test space (see Fix et al. [65] and Blum/Dobrowolski [47]. All these techniques cause significant overheads and it is not always easy to incorporate them into existing software. Here, we shall discuss another approach based on Richardson extrapolation, which is also capable to recover the full second-order accuracy of the method, but uses only the standard shape functions and quasi-uniform meshes.

The underlying idea of the extrapolation algorithm described below is a splitting of the error of the form

$$u - R_h u = (s - R_h s) + (U - R_h U), \tag{6.6.167}$$

where $s - R_h s$ represents the "pollution" part and $U - R_h U$ the "smooth" part of the projection error. For $U - R_h U$ one has the optimal order of convergence $\mathcal{O}(h^2)$ at interior nodes, while the pollution term is assumed to be of the form

$$(s - R_h s)(x) = A(x) h^{2\alpha} + \mathcal{O}(h^2), \tag{6.6.168}$$

at interior points $x \in \Omega_0 \subset \Omega$ and therefore may be eliminated by extrapolation.

$$u_h^* := (2^{2\alpha} - 1)^{-1} \big(2^{2\alpha} R_{h/2} u - R_h u \big) = u + \mathcal{O}(h^2). \tag{6.6.169}$$

The presence of such an expansion is suggested by the error estimate (6.6.166) and by a result of Nitsche [98], which asserts that

$$(u - R_h u)(x) = A(h) s_{-1}(x) + U - R_h U \tag{6.6.170}$$

where $s_{-1}(x) \sim r^{-\pi/\omega}$ is the first "dual" singular function corresponding to the corner point z and the "regular" part $U - R_h U$ of the error is of the optimal order $\mathcal{O}(h^2)$. The results of one extrapolation step applied for the slit domain Ω_2 (see Fig. 6.11) with $2\alpha = 1$ are listed in Table 6.11.

Table 6.11: Demonstration of the pollution effect for the slit domain and the result of h-extrapolation, 1% error level indicated by boldface ($2\alpha = 1,\ h_i = 2^{-i}$).

i	$\\|u^{(1)} - R_h u^{(1)}\\|_{\infty;\Omega_0}$	$\\|u^{(2)} - R_h u^{(2)}\\|_{\infty;\Omega_0}$	$\\|u^{(2)} - u_h^*\\|_{\infty;\Omega_0}$
3	5.5 (-1)	2.1 (-1)	
4	**1.3 (-2)**	8.6 (-2)	2.80 (-2)
5	3.3 (-3)	3.9 (-2)	**1.49 (-2)**
6	8.8 (-4)	1.9 (-2)	3.20 (-3)
7	1.4 (-4)	**9.5 (-3)**	7.00 (-4)
1% error	$\sim$ 200 nodes	$\sim$ 13.000 nodes	$\sim$ 800 nodes

In the presence of several critical boundary points z_n, $n = 1, \dots, N$, with corresponding interior angles $\omega_n \in (\pi, 2\pi]$ the extrapolation approach is based on an asymptotic error expansion of the form

$$R_h u(x) = u(x) + \sum_{n=1}^{N} A_n K_{n;1}(u) s_{n;-1}(x) h_n^{2\alpha_n} + \mathcal{O}(h^2 L(h)), \tag{6.6.171}$$

on meshes, which are in a certain sense uniform (stretching invariant and symmetric) in the neighborhood of the critical points. Here, $\alpha_n = \omega_n/\pi$ is the exponent of the leading singularity in u at the point z_n. The expansion coefficients contain the corresponding "stress intensity factor" $K_{n;1}(u)$, the dual singular function $s_{n;-1}$, and certain constants A_n depending on the characteristics of the discretization. The global mesh size is h, while h_n refers to the local mesh size in a neighborhood of the corner point z_n. The remainder term becomes singular as $x \to z_n$, but is integrable over Ω. Such an expansion has been derived for the model problem (6.6.163) in Blum/Rannacher [50]; see also the survey article [105]). The highly technical proof cannot be given in this text. Related results for certain finite difference discretizations on rectangular domains have been obtained by Zenger/Gietl [137].

Remark 6.15: The proof of the error expansion (6.6.171) is based on the following detailed information on the regularity of the solution u of the model problem (6.5.131) on a domain with reentrant corners as described above. The solution admits a representation of the form

$$u = \sum_{n=1}^{N} \sum_{i=1}^{I_n} K_{n;i}(u) s_{n;i} + U, \tag{6.6.172}$$

with a "smooth" remainder term $U \in C^2(\bar{\Omega})$ and the so-called "stress intensity" factors $K_{n;i}(u)$. The "singular" functions have the form (using polar coordibates (r_n, θ_n) at the corner point z_n)

$$s_{n;i}(x) = (|i|\pi)^{-1/2} r_n^{\alpha_{n;i}} \sin(\alpha_{n;i}\theta_n), \quad \alpha_{n;i} = i\pi/\omega_n \;\; (i \in \mathbb{Z}).$$

For the stress intensity factors, which play an important role in the analysis of crack propagation in elastic materials, one has the representation (see Blum/Dobrowolski [47]

$$K_{n;i}(u) = (f, \tau_n s_{n;-i})_\Omega - (u, \partial_n[\tau_n s_{n;-i}])_{\partial\Omega} + (u, \Delta[\tau_n s_{n;-i}])_\Omega,$$

where $\tau_n = \tau_n(r_n)$ are smooth cut-off functions separating the several corner points of $\partial\Omega$. Representations of the type (6.6.172) are known for very general linear and also some weakly nonlinear elliptic problems (see, e. g., Grisvard [8], Kondratiev [13], and Blum/Rannacher [49]).

Remark 6.16: The limitation to the order $\mathcal{O}(h^2)$ in the expansion (6.6.171) is natural in considering a second-order discretization. One might seek to carry the expansion further to higher order in h by using elements of higher polynomial degree or, with linear elements, by using blockwise uniform meshes. However, there seems to be the upper limit with order $r := \min_n\{4\alpha_{n;1}\}$ for localizing the pollution effects to the neighborhoods of the critical corner points.

Remark 6.17: Expansion results similar to (6.6.171) also hold for the approximation of eigenvalues (and other globally defined quantities such as, e. g., torsion moments and stress intensity factors, on domains with reentrant corners (see Blum/Rannacher [51]),

$$\lambda_h = \lambda + \sum_{n=1}^{N} C_n h_n^{2\alpha_n} + \mathcal{O}(h^2). \tag{6.6.173}$$

For proving this it is important to know that the remainder term in the error expansion (6.6.171) is in $L^1(\Omega)$, as a function of x, uniformly in h.

Remark 6.18: In the case of many critical boundary points the naive use of Richardson extrapolation on the basis of the error expansion (6.6.171) requires the solution of the discrete problem for $N+1$ successively refined meshes. In general this would cause an unacceptable amount of computational work. However, since in the expansion (6.6.171) the effect of the different critical points z_n is localized through the appearance of the local mesh sizes h_n, one may work with meshes, which are refined only in the neighborhoods $\bar{\Omega}_n \subset \bar{\Omega}$ of the points z_n. Therefore, by properly adjusting the local mesh refinements to the exponents $2\alpha_n$, one extrapolation step can simultaneously eliminate all pollution terms. This procedure requires only the solution of the discrete problem on two different meshes (for details see [50] and [105]). For the extrapolation procedure one needs to know only (approximate) values of the "singular" exponents α_n occuring in the expansion (6.6.171). These are available for many linear and also some weakly nonlinear problems in Continuum Mechanics (e. g., Lamé-Navier system, Kirchhoff and von Kármán plate models, Stokes and Navier-Stokes system).

6.6.4 Extrapolation in the FE method for parabolic problems

To come (see [108]).

6.7 Exercises

Exercise 6.1: Let the abstract differential problem $Lu = f$ be approximated by a (low-order) stable discretization

$$L_h u_h = r_h f,$$

i. e., $\sup_{h>0} \|L_h^{-1}\| < \infty$. Further, let

$$\hat{L}_h \hat{u}_h = \hat{r}_h f,$$

be another discretization of higher-order but not necessarily stable. Then, defect correction may be used in the following form:

$$d_h := \hat{L}_h u_h - \hat{r}_h f, \quad L_h \delta u_h = -d_h, \quad u_h^{(1)} := u_h + \delta u_h.$$

Show, that for iterating this process, one gets the error estimate

$$\|u_h^t - \hat{r}_h u\| \le q_h^t \|u_h - \hat{r}_h u\| + \frac{q_h}{1-q_h} \|L_h^{-1}\| \|\hat{L}_h \hat{r}_h u - \hat{r}_h f\|, \quad t \ge 1,$$

provided that $q_h := \|I - L_h^{-1}\hat{L}_h\| < 1$. *Remark.* Notice that due to the possible instability of $\hat{L}_h$ in general $\|I - L_h^{-1}\hat{L}_h\| > 1$ so that in this case the above estimate cannot not be taken as justification for the use of defect correction.

Exercise 6.2: Let be given $n+1$ pairwise different points $x_i \in \mathbb{R}^1, i = 0, 1, \dots, n$, and the corresponding $n+1$ Lagrange base polynomials

$$L_i^{(n)}(x) = \prod_{j=0, j\ne i}^{n} \frac{x - x_j}{x_i - x_j}, \quad i = 0, \dots, n.$$

Show that the polynomials $\{L_i^{(n)}, i = 0, \dots, n\}$, form a basis of the polynomial space P_n (vector space of all polynomials of degree less or equal n), and that the following relations hold:

$$i) \qquad \sum_{i=0}^{n} L_i^{(n)}(x) = 1, \quad x \in \mathbb{R}^1,$$

$$ii) \qquad \sum_{i=0}^{n} x_i^k L_i^{(n)}(0) = 0, \quad k = 1, \dots, n,$$

$$iii) \qquad \sum_{i=0}^{n} x_i^{n+1} L_i^{(n)}(0) = (-1)^n \prod_{i=0}^{n} x_i.$$

(Hint: One may use the uniqueness of the Lagrange interpolation polynomial and the representation of the error in the Lagrange interpolation.)

Exercise 6.3: For the function $f(x) = \cosh(x)$ the following table of values is given:

x	$f(x)$
0.52	$1,1382741$
0.56	$1,1609408$
0.60	$1,1854652$
0.64	$1,2118867$
0.68	$1,2402474.$

Determine by extrapolation of a suitable difference quotient an best-possible approximation to the derivative value $f'(0.6) = 0,63665358\dots$.

Exercise 6.4: Which one the index sequences

$$(i) \quad n_i = 2i - 1,\; i \in \mathbb{N}, \qquad (ii) \quad n_i = 3^i,\; i \in \mathbb{N}, \qquad (iii) \quad n_i = i^2,\; i \in \mathbb{N},$$

for step sizes $h_i = h/n_i$ is admissible for the stable extrapolation to the limit $h = 0$?

Exercise 6.5: a) Recall the basic principles of numerical stability theory for one-step methods in solving ODE (provided in class).

b) Write the modified midpoint rule of Gragg (with an explicit Euler step as starting procedure) for the step size h/N with $N = 2$ as (explicit) Runge-Kutta method with step size h.

c) What is the order of this one-step scheme?

d) The simple midpoint rule has a trivial stability region. Verify that the above Runge-Kutta method has a nontrivial stability region, especially, its stability interval is $\mathrm{SI} = [-3.1, 0]$. This demonstrates the stabilization effect of Gragg's averaging procedure.

Exercise 6.6: Show that the implicit Euler scheme

$$y_n = y_{n-1} + h_n f(t_n, y_n), \quad n \geq 1, \quad y_0 = u_0.$$

admits an asymptotic error expansion of the form

$$y_n = u(t_n) + h e(t_n) + \mathcal{O}(h^2).$$

(Hint: Adapt the corresponding argument given in class for the explicit Euler scheme.)

Exercise 6.7: The application of extrapolation for a *stiff* IVP requires the use of an A-stable formula as base scheme, e.g. the implicit Euler scheme (or the trapezoidal rule)

$$y_n = y_{n-1} + h_n f(t_n, y_n), \quad n \geq 1, \quad y_0 = u_0.$$

a) Describe for this case the extrapolation process. Which order is achieved if the whole Bulirsch sequence $\{2,4,6,8,12,16\}$ is used?

b) Investigate whether the resulting implicit one-step scheme ($m=1$) is again A-stable. For this consider the simplest case of just one extrapolation step and determine the amplification factor $\omega(z)$ only for $z\in\mathbb{R}$ (stability interval).

Exercise 6.8: Verify that for the one-dimensional Poisson problem

$$-u''(x)=f(x)\quad\text{in }\ \Omega=(0,1),\quad u(0)=u(1)=0,$$

the Ritz projection $R_h: H_0^1(\Omega)\to V_h^{(1)}$ coincides with the nodal interpolation, i. e., $R_hu=I_hu$. What does this mean for the question of asymptotic error expansions?

Exercise 6.9: Prove the following identity for functions $u\in C^4(I)$ and their nodal interpolant $I_hu\in P_1(I)$ on the interval $I=[0,h]$ (Euler-MacLaurin formula):

$$\int_I (u-I_hu)\,ds=-\frac{h^2}{12}\int_I u''(s)\,ds+\frac{h^4}{24}\int_I \sigma(s)u^{(4)}(s)\,ds\,,$$

where $\sigma(s)=h^{-2}s^2(1-h^{-1}s)^2$. (Hint: Transform the integrals to the interval $[0,1]$ and use integration by parts.)

Exercise 6.10: Use the asymptotic expansion results for the Ritz projection on three-directional meshes derived in class to show that:

a) In the (admittedly unrealistic) case that the solution $u\in H_0^1(\Omega)$ happens to be a *cubic* polynomial its Ritz projection coincides with its interpolant, i. e., $R_hu=I_hu$.

b) If the mesh consists of equilateral triangles the Ritz projection is of 4th-order accuracy at nodal points, i. e., there holds

$$(u-R_hu)(P)=\mathcal{O}(h^4L(h)).$$

(Hint: On an equilateral mesh there holds $\lambda_1=\lambda_2=\lambda_3$ and $D_1+D_2+D_3=0$.)

Exercise 6.11: Carry the asymptotic expansion result for the consistency error $(\nabla(u-I_h),\nabla\varphi_h)$ derived in class to higher-order $n=3$, i. e.,

$$(\nabla(u-I_hu),\nabla\varphi_h)=h^2(\nabla e^{(1)},\nabla\varphi_h)_\Omega+h^4(\nabla e^{(2)},\nabla\varphi_h)_\Omega+h^6\rho_h(u;\varphi_h),$$

where $e^{(1)}$ and $e^{(2)}$ are determined as solutions of certain auxiliary problems.

Exercise 6.12: Let $\mathbb{T}_h$ be a three-directional mesh on which the Ritz projection R_hu admits an asymptotic error expansion of the form

$$R_hu(P)=u(P)+h^2e^{(1)}(P)+\mathcal{O}(h^4L(h)),$$

at interior nodal points $P \in \Omega'' \subset\subset \Omega$. Let the finer mesh $\mathbb{T}_{h/2}$ be constructed from $\mathbb{T}_h$ by cutting each triangle into four congruent subtriangles. Construct by extrapolating $R_h u$ and $R_{h/2} u$ an approximation $\bar{u}_h \in V_{h/2}$, which satisfies

$$(\bar{u}_h - u)(P) = \mathcal{O}(h^4 L(h)),$$

at interior nodal points of the coarser mesh $\mathbb{T}_h$. Why does this improvement generally not hold in interior nodal points of the finer mesh $\mathbb{T}_{h/2}$?

Exercise 6.13: Let $\mathbb{T}_h$ be a shape- and size-regular triangulation of the domain $\Omega \subset \mathbb{R}^2$ and $V_h \subset H_0^1(\Omega)$ the corresponding space of "linear" finite element functions. Further let $z \in \Omega$ be an arbitrary but fixed point and $T_z \in \mathbb{T}_h$ a triangle containing z. Show that there exists a function δ_z^h (regularized Dirac function) with support $\operatorname{supp}(\delta_z^h) \subset T_z$ and $\|\delta_z^h\|_{L^\infty} \le ch^{-2}$ uniformly w.r.t. z and h, such that

$$(\varphi_h, \delta_z^h) = \varphi_h(z), \quad \varphi_h \in V_h.$$

(Hint: Consider the associated reference unit triangle $\hat{T}$.)

Exercise 6.14: On polygonal domains $\Omega \subset \mathbb{R}^2$ there holds the following "L^1-trace inequality":

$$\int_{\partial\Omega} |v|\, ds \le c\big\{\|v\|_{L^1(\Omega)} + \|\nabla v\|_{L^1(\Omega)}\big\}, \quad v \in C^1(\bar{\Omega}).$$

Use this inequality to derive its cellwise version on each triangle of a regular triangulation $\mathbb{T}_h$ of $\bar{\Omega}$:

$$\int_{\partial T} |v|\, ds \le c\big\{h^{-1}\|v\|_{L^1(T)} + \|\nabla v\|_{L^1(T)}\big\}, \quad T \in \mathbb{T}_h, \;\; v \in C^1(T).$$

(Hint: Consider the associated reference unit triangle $\hat{T}$.)

Exercise 6.15: Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, $V := H_0^1(\Omega)$ and $V_h \subset V$ the usual subspaces of "linear" finite elements on a quasi-uniform family $\{\mathbb{T}_h\}_{h>0}$ of triangulations of $\bar{\Omega}$. Derive for the Ritz projection $R_h : V \to V_h$ the L^2-stability estimate

$$\|R_h v\| \le c\|v\| + ch\|\nabla v\|, \quad v \in V \cap H^2(\Omega),$$

and use this to prove the optimal-order L^2-error estimate

$$\|u - R_h u\| \le ch^2\|u\|_{2,2}, \quad v \in V \cap H^2(\Omega).$$

Do these results also hold true on meshes not necessarily satisfying the uniform size condition? (Hint: Use the standard "duality argument".)

Exercise 6.16: In class the "real" Green's function $g_x(y)$ has been used, for which in 2D, we know the estimate

$$|g_x(y)| \le c\,|\ln(|x-y|)| + 1.$$

Use this to show that on any circle $B_\rho(x)$ with midpoint x and radius $\rho \le 1$, there holds the estimate

$$|B(x)|^{-1} \int_{B_\rho(x)} |g_x(y)|\, dy \le cL(\rho),$$

with a constant c independent of x and ρ. (Hint: Rewrite the integral using polar coordinates.)

Exercise 6.17: Modify the proof of the L^1 estimates for the regularized Green's function given in class for mesh families $\{\mathbb{T}_h\}_{h\in\mathbb{R}_+}$, which are not necessarily "strongly size-regular" but only "polynomial size-regular", in the following sense:

$$h_{\min} \ge ch_{\max}^\alpha \quad \text{for some } \alpha \ge 1.$$

(Hint: Recall the modifications of the argument mentioned in class and observe that $L(h_{\min}) \le L(h_{\max}^\alpha) \le c\alpha L(h)$.)

Exercise 6.18: Let $(\mathbb{T}_h)_h$ be a quasi-uniform (structur-, shape- and size-uniform) family of triangulations of a convex polygonal domain $\Omega \subset \mathbb{R}^2$ and V_h the corresponding spaces of *linear* finite elements. Show that for the Ritz-projection $R_h u \in V_h$ of a functions $u \in H_0^1(\Omega) \cap H^2(\Omega)$ there holds the reduced-order L^∞-error estimate

$$\|u - R_h u\|_{L^\infty} \le ch\|u\|_{H^2}.$$

(Hint: One may use the nodal interpolant $I_h u \in V_h$, an "L^∞/L^2 inverse property" of finite elements, the standard interpolation estimate and the standard L^2-error estimate.)

Exercise 6.19: In class the following a priori bounds have been shown for the regularized Green's function:

$$\|g_z^h\|_{L^\infty} \le cL(h),$$
$$\|\nabla g_z^h\| + \|\sigma\nabla^2 g_z^h\| \le cL(h)^{1/2},$$

where $\sigma(x) := (|x-z|^2 + h^2)^{1/2}$ and $L(h) := \ln(h) + 1$.

i) Prove the pointwise bound

$$|g_z^h(x)| \le cL(\sigma(x)), \quad x \in \Omega.$$

ii) Prove the following log-free modification of the a priori bound:

$$\|\sigma^\varepsilon \nabla g_z^h\| + \|\sigma^{1+\varepsilon}\nabla^2 g_z^h\| \le c(\varepsilon),$$

for any fixed $\varepsilon > 0$.

Exercise 6.20: Prove the lower bound

$$g_z^h(z) \ge cL(h).$$

(Hint: Use the splitting for the exact Green's function $G_x(y) = -\frac{1}{2\pi}\ln(x-y) + \psi(x,y)$ with a bounded part $\psi(x,y)$.)

Exercise 6.21: Let g_P be the Greens's function of the Laplacian corresponding to some nodal point P of the triangulation $\mathbb{T}_h$ of $\bar{\Omega}$. Further, let $\tilde{I}_h g_P \in V_h$ be the corresponding nodal interpolant, which is modified at the singular point P according to

$$\tilde{I}_h g_P(P) = \bar{g}_P, \qquad \bar{g}_P := |B_P|^{-1}\int_{B_P} g_P\,dx, \qquad B_P := \cup\{T \in \mathbb{T}_h \,|\, P \in T\}.$$

Prove the error estimate

$$\|\nabla(g_P - \tilde{I}_h g_P)\|_{L^1(\Omega)} = \mathcal{O}(hL(h)).$$

(Hint: One may use the estimates $\|\nabla^2 g_P\|_{L^1(\Omega\setminus B_P)} = \mathcal{O}(L(h))$, $\|\nabla g_P\|_{L^1(B_P)} = \mathcal{O}(h)$, and $|g_P(x)| \le c(|\ln(|x-P|) + 1)$.)

Exercise 6.22: Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, Ω_0 a subregion having positive distance to the corner points of $\partial\Omega$, and $\mathbb{T}_h$ a (form- and shape) regular triangulation of $\bar{\Omega}$. Further let $\Gamma \subset \Omega_0$ be a straight line consisting of sides of triangles $T \in \mathbb{T}_h$ and P a nodal point having positive distance to the endpoints a, b of Γ and to the corner points of $\partial\Omega$ (One may make a sketch of this situation.).
For the corresponding Green's function g_P prove the estimate

$$\int_\Gamma (R_h g_P - g_P)\psi\,ds = \mathcal{O}(h^2 L(h)),$$

with any smooth weight function ψ vanishing near a and b. To this end, one may proceed as follows: Let $v \in H_0^1(\Omega)$ be the solution of the dual problem

$$(\nabla\varphi, \nabla v) = \int_\Gamma \varphi\psi\,ds \quad \forall\varphi \in H_0^1(\Omega),$$

which satisfies $v \in H^2(\Omega\setminus\Gamma) \cap C^{2+\alpha}(B_P)$ for a suitable mesh region B_P surrounding P. For this, we have the error estimates

$$\|v - R_h v\| + h\|\nabla(v - R_h v)\| = \mathcal{O}(h^2),$$
$$\|\nabla(v - R_h v)\|_{L^\infty(B_P)} = \mathcal{O}(h),$$

depending on the data ψ. One can justify setting $\varphi = R_h g_P - g_P$ in the dual problem to obtain (with the modified nodal interpolant $\tilde{I}_h g_P$ defined in the preceding exercise)

$$\int_\Gamma (R_h g_P - g_P)\psi\,ds = (\nabla(R_h g_P - g_P), \nabla v) = (\nabla(\tilde{I}_h g_P - g_P), \nabla(v - R_h v)).$$

Therefore, it remains to show that

$$(\nabla(\tilde{I}_h g_P - g_P), \nabla(v - R_h v)) = \mathcal{O}(hL(h)),$$

which is the task of this exercise. (Hint: Use an appropriate splitting of the domain Ω and suitable local error estimates.)

Exercise 6.23: Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain (say the unit square), $\mathbb{T}_h$ be a uniform triangulation of $\bar{\Omega}$, and the solution of the model problem satisfy $u \in H_0^1(\Omega) \cap C^{4+\alpha}(\bar{\Omega})$. In class it has been shown that defect correction with *quadratic* finite elements leads to a global error behavior like

$$\|u_h^* - I_h u\|_{L^\infty} = \mathcal{O}(h^3 L(h)).$$

a) Formulate the defect correction method with *cubic* Lagrange-finite elements (alternatively *cubic* Hermite-finite elements) defined on suitable patches of triangles of $\mathbb{T}_h$.

b) Show that this higher-order defect correction yields the error behavior

$$\|u_h^* - I_h u\|_{L^\infty(\Omega_0)} = \mathcal{O}(h^4 L(h)),$$

on subdomains $\Omega_0 \subset \Omega$ having positive distance from the corner points of $\partial\Omega$. (Hint: Follow, with appropriate modifications, the argument used in class for defect correction with *quadratic* finite elements.)

Exercise 6.24: Consider the approximation of the model problem on the unit square $\Omega = (0,1)^2$ with *linear* finite elements on uniform meshes $\mathbb{T}_h$ of width h. Compare the complexity of

- Richardson extrapolation,
- defect correction with *cubic* Lagrange-finite elements,
- direct use of *cubic* Lagrange-finite elements,

all methods being of order $\mathcal{O}(h^4 L(h))$, with respect to the number of nonzero elements in the corresponding system matrices, which is relevant for the use of iterative methods, e. g., the CG-method and the multigrid method.

Exercise 6.25: Give short answers to the following questions:

1. Which one of the index sequences

$$(i) \quad n_i = 2i-1,\ i \in \mathbb{N}, \qquad (ii) \quad n_i = 3^i,\ i \in \mathbb{N}, \qquad (iii) \quad n_i = i^2,\ i \in \mathbb{N},$$

   for step sizes $h_i = h/n_i$ is admissible for the stable Richardson extrapolation to the limit $h = 0$?

2. What kind of asymptotic error expansions are valid for the explicit and implicit Euler scheme in solving ODE systems?

3. Why is in solving ODE systems the original 2-step midpoint rule not suitable as base method for Richardson extrapolation?

4. Extrapolation uses the solutions u_h and $u_{h/2}$ on two successively refined meshes $\mathbb{T}_h$ and $\mathbb{T}_{h/2}$, respectively. On which mesh does this method yield improved approximations?

5. Let $a(\cdot,\cdot)$ be a scalar product on the Sobolev space $H_0^1(\Omega)$. What is the definition of the corresponding Ritz projection $R_h : H_0^1(\Omega) \to V_h$ onto some finite element subspace $V_h \subset H_0^1(\Omega)$?

6. What is meant by the term "Galerkin orthogonality"?

7. Which properties must a family of "triangulations" (decompositions into triangles) $\{\mathbb{T}_h\}_{h>0}$ of a polygonal domain $\Omega \subset \mathbb{R}^2$ possess in order to be called "quasi-uniform"?

8. When is a triangulation $\mathbb{T}_h$ of a domain $\Omega \subset \mathbb{R}^2$ called a) "uniform (three-directional)", b) "blockwise uniform", and c) "blockwise parametrically uniform"?

9. Consider the usual error expansion of the Ritz projection on a uniform mesh covering a (convex) polygonal domain. By what difficulty is the order of the remainder term in general limited to $\mathcal{O}(h^3)$?

10. Why is in "defect correction" by higher-order interpolation with defect computation using quadratic elements the achievable order in general limited to $\mathcal{O}(h^3)$?

Exercise 6.26: Sketch the proof of the (optimal-order) error estimate

$$\|u - R_h u\|_{L^\infty} \le ch\|u\|_{H^2}$$

for the FE-Galerkin approximation of the model problem considered in class with *linear* elements on quasi-uniform triangulations $\mathbb{T}_h$ of a convex polygonal domain $\Omega \subset \mathbb{R}^2$. Why are the quasi-uniformity of the mesh and the convexity of the domain essential for this result?

Exercise 6.27: Determine the orders to be expected for the approximation of the usual model problem a) on a slit domain (interior angle $\omega = 2\pi$) and b) on an L-shaped domain (interior angle $\omega = \frac{3}{2}\pi$) in the following expressions:

$$\|\nabla(u - R_h u)\|_{L^2} = \mathcal{O}(h^?), \qquad \|u - R_h u\|_{L^\infty(\Omega_0)} = \mathcal{O}(h^?),$$

where $\Omega_0 \subset \Omega$ is a subdomain with positive distance to the corner points of $\partial\Omega$.

Exercise 6.28: On a domain Ω with a reentrant corner with interior angle $\omega \in (\pi, 2\pi]$ the Ritz projection admits an asymptotic error expansion of the form

$$R_h u(P) = u(P) + h^{2\alpha} e(P) + \mathcal{O}(h^2 L(h)),$$

with $\alpha = \pi/\omega$ at interior nodal points P. How does in this case one step of extrapolation for achieving the order $\mathcal{O}(h^2 L(h))$ look like?

Bibliography

[1] R. Rannacher: *Numerik 0: Einführung in die Numerische Mathematik*, Heidelberg University Publishing, 2017,
`http://heiup.uni-heidelberg.de/catalog/book/206`.

[2] R. Rannacher: *Numerik 1: Numerik Gewöhnlicher Differentialgleichungen*, Heidelberg University Publishing, 2017,
`http://heiup.uni-heidelberg.de/catalog/book/258`.

[3] R. Rannacher: *Numerik 2: Numerik partieller Differentialgleichungen*, Lecture Notes, Heidelberg University,
`http://numerik.uni-hd.de/∼lehre/notes/`

[4] R. Rannacher: *Numerik 3: Probleme der Kontinuumsmechanik und ihre Finite-Elemente-Approximation*, Lecture Notes, Heidelberg University,
`http://numerik.uni-hd.de/∼lehre/notes/`

(I) General References on Functional Analysis, ODE, PDE and their Numerical Solution

[5] R. A. Adams: *Sobolev Spaces*, Academic Press, New York, 1975.

[6] P. G. Ciarlet and J.L. Lions: *Handbook of Numerical Analysis Volume II, Finite Element Methods I, and Volume IV, Finite Element Methods II*, North-Holland: Amsterdam, 1991.

[7] G. P. Galdi: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. 1: Linearized Steady problems, Vol. 2: Nonlinear Steady Problems*, Springer: Berlin-Heidelberg-New York, 1998.

[8] P. Grisvard: *Elliptic Problems in Nonsmooth Domains*, Pitman Publishing, Marshfield, Massachusetts, 1985.

[9] P. M. Halmos: *Finite Dimensional Vector Spaces*, Springer, 1974.

[10] G. Hellwig: *Partial Differential Equations. An Introduction*,
B.G. Teubner, Stuttgart, 1977.

[11] J. Joos: *Partial Differential Equations*, Springer 2013.

[12] T. Kato: *Perturbation Theory for Linear Operators*, Springer, 2nd ed., 1980.

[13] V. A. Kondratiev: *Boundary Value Problems for Elliptic Equations in Domains with Conical or Angular Points*, Trans. Moscow Math. Soc. 16, 227–313 (1967).

[14] P. Lax: *Functional Analysis*, Wiley-Interscience, 2002.

[15] M. Renardy, R. Rogers: *An Introduction to Partial Differential Equations*,
Springer 1993.

[16] F. Riesz and B. Sz.-Nagy: *Functional Analysis*, Dover Publications, 1990.

[17] W. Rudin: *Functional Analysis*, McGraw-Hill Science, 1991.

[18] M. Schechter: *Principles of Functional Analysis*, AMS, 2nd ed., 2001.

[19] J. Stoer and R. Bulirsch: *Einf"uhrung in die Numerische Mathematik*, Teil I (1972, J. Stoer), Teil II (1973, J. Stoer und R. Bulirsch); Springer (new edition).

[20] W. R. Strauss: *Partial Differential Equations: An Introduction*, John Wiley 1992.

[21] R. Temam: *Navier-Stokes Equations. Theory and numerical analysis.* North Holland: Amsterdam, 1987.

[22] A. Tveito and R. Winther: *Introduction to Partial Differential Equations: A Computational Approach*, Springer, 1998.

[23] J. Wloka: *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987.

[24] K. Yosida: *Functional Analysis*, Springer, 6th ed., 1980.

(II) General References on the Finite Element Method

[25] T- Apel: *Anisotropic Finite Elements: Local Estimates and Applications*, B.G.Teubner: Stuttgart-Leipzig, 1999.

[26] O. Axelsson and V. A. Barker: *Finite Element Solution of Boundary Value Problems, Theory and Computation*, Academic Press, 1984.

[27] W. Bangerth and R. Rannacher: *Adaptive Finite Element Methods for Differential Equations*, Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel 2003.

[28] D. Braess: *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, Springer 2007 (3rd edition).

[29] S. C. Brenner, L. R. Scott: *The Mathematical Theory of Finite Element Methods*, Springer 1994.

[30] F. Brezzi and M. Fortin: *Mixed and Hybrid Finite Element Methods*, Springer 1991.

[31] P. G. Ciarlet: *The Finite Element Method for Elliptic Problems*, North-Holland 1978.

[32] K. Eriksson, D. Estep, P. Hansbo, C. Johnson: *Computational Differential Equations*, Cambridge University Press 1996.

[33] V. Girault and P.-A. Raviart: *Finite Element Methods for the Navier-Stokes Equations.* Springer: Berlin-Heidelberg-New York, 1986.

[34] W. Hackbusch: *Elliptic Differential Equations. Theory and Numerical Treatment*, Springer, 1992.

[35] C. Johnson: *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge University Press 1987.

[36] A. Quarteroni and A. Valli: *Numerical Approximation of Partial Differential Equations*, Springer, 1994.

[37] G. Strang and G. J. Fix: *An Analysis of the Finite Element Method*, Prentice-Hall, 1973.

(III) Special References

[38] M. Asadzadeh, A. H. Schatz, and W.Wendland: *A non-standard approach to Richardson extrapolation in the finite element method for second order elliptic problems*, Math. Comp. 79, 1951-1973 (2009).

[39] M. Asadzadeh, A. H. Schatz, and W.Wendland: *Asymptotic error expansions for the finite element method for second order elliptic problems in $\mathbb{R}^N, N \geq 2$. I: Local interior expansions*, SIAM J. Numer. Anal. 48, 2000-2017 (2010).

[40] I. Babuška: *Finite element methoid for domains with corners*, Computing 6, 264-273 (1970).

[41] R. Becker and M. Braack: *Solution of a benchmark problem for natural convection at low Mach number*, Int. J. Thermal Sciences 41, 428–439 (2002).

[42] R. Becker, C. Johnson, and R. Rannacher: *Adaptive error control for multigrid finite element methods*, Computing 55, 271-288 (1995).

[43] R. Becker, H. Kapp, and R. Rannacher: *Adaptive finite element methods for optimal control of partial differential equations: Basic concepts*, SIAM J. Control Optim. 39, 113132 (2000).

[44] R. Becker and R. Rannacher: *An optimal control approach to a posteriori error estimation in finite element methods*, Acta Numerica 2000 (A. Iserles, ed.), pp. 1-102, Cambridge University Press, 2001.

[45] H. Blum: *On Richardson extrapolation for linear finite elements on domains with reentrant corners*, Z. Angew. Math. Mech. 29, 51-353 (1987).

[46] H. Blum: *Asymptotic error expansion and defect correction in the finite element method*, Habilitationsschrift, Univ. Heidelberg, 1990.

[47] H. Blum and M. Dobrowolski: *On finite element methods for elliptic equations on domains with corners*, Computing 28, 53–63 (1982).

[48] H. Blum, Q. Lin, and R. Rannacher: *Asymptotic error expansion and Richardson extrapolation for linear finite elements*, Numer. Math. 49, 11–37 (1986).

[49] H. Blum and R. Rannacher: *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Meth. Appl. Sci. 2, 556-581 (1980).

[50] H. Blum and R. Rannacher: *Extrapolation techniques for reducing the pollution effect of reentrant corners in the finite element method*, Numer. Math. 52, 539-564 (1988).

[51] H. Blum and R. Rannacher: *Finite element eigenvalue computation on domains with reentrant corners using Richardson extrapolation*, J. Comput. Math. 8, 321-332 (1990).

[52] K. Böhmer: *Asymptotic expansions for the discretization error in linear elliptic boundary value problems for general region*, Math. Z. 177, 235-255 (1982).

[53] K. Böhmer, P.W. Hemker, and H.J. Stetter: *The defect correction approach*, in "Defect Correction Methods. Theory and Applications", K. Böhmer and H.J. Stetter, eds., pp. 1–32, Computing Suppl. 5, Springer, 1984.

[54] M. Braack and A. Ern: *A posteriori control of modeling errors and discretization errors*, Muliscale Model. Simul. 1, 221–238 (2003).

[55] M. Braack et al.: *Modelling of natural convection flows with large temperaure differences: a benchmark problem for low Mach number solvers. part 1 and part 2.*, M2AN 39, 609-616, 616–621 (2005).

[56] M. Braack and Th. Richter: *Solutions of 3D Navier-Stokes benchmark problems with adaptive finite elements*, Computers and Fluids 35, 372–392 (2006).

[57] C. Carstensen and R. Klose: *A posteriori finite element error control for the p-Laplace problem*, SIAM J. Sci. Comput. 25, 792–814 (2003).

[58] C. M. Chen and Q. Lin: *Extrapolation of finite element approximations in a rectangular domain*, J. Comput. Math 7, 227-233 (1989).

[59] H. Chen and R. Rannacher: *Local error expansions and Richardson extrapolation for the streamline diffusion finite element method*, East-West J. Numer. Math. 1, 253-265 (1993).

[60] Y. H. Ding and Q. Lin: *Finite element expansion for variable coefficient elliptic problems*, Systems Sci. Math. Sci. 2, 54-69 (1989).

[61] M. Dobrowolski and R. Rannacher: *Finite element methods for nonlinear elliptic systems of second order*, Math. Nachr. 95, 155-172 (1980).

[62] W. Dorfler: *A convergent adaptive algorithm for Poissons equation*, SIAM J, Numer. Anal. 33, 1106–1124 (1996).

[63] R. Frank, R. H. Nochetto, and J. Wang: *Sharp maximum norm error estimates for finite element approximations of the Stokes problem in 2-d*, Math. Comp. 51, 491–506 1988).

[64] Weinan E, H. C. Huang, and W. M. Han: *Error analysis of local refinements of polygonal domains*, J. Comput. Math. 5,89–94 (1987).

[65] G. J. Fix, S. Gulati, and G. I. Wakoff: *On the use of singular functions with finite element approximations*, J. Comp. Phys. 13, 2009–238 (1973).

[66] R. Frank, J. Hertling, and J. O. Monnet: *The application of iterated defect correction to variational methods for elliptic boundary value problems*, Computing 30, 121–135 (1983).

[67] J. Frehse and R. Rannacher: *Eine L^1-Fehlerabschätzung für diskrete Grundlösungen in der Methode der finiten Elemente*, Bonn. Math. Schr. 89, 92-114 (1976).

[68] J. Frehse and R. Rannacher: *Asymptotic L^∞-error estimates for linear finite element approximations of quasi-linear boundary value problems*, SIAM J. Numer. Anal. 15, 418-431 (1978).

[69] C. I. Goldstein: *Variational crimes and L^∞ error estimates in the finite elemenrt method*, Math. Comp. 35, 1131–1157 (1980).

[70] P. M. Hanson and J. E. Walsh: *Asymptotic theory of the global error and some techniques of error estimation*, Numer. Math. 45, 51–74 (1984).

[71] V. Heuveline and R. Rannacher: *A posteriori error control for finite element approximations of elliptic eigenvalue problems*, Advances in Comput. Math. 15, 1-32 (2001).

[72] V. Heuveline and R. Rannacher: *Duality-based adaptivity in the hp-finite element method*, J. Numer. Math. 11, 95–113 (2003).

[73] J. G. Heywood and R. Rannacher: *Finite element approximation of the nonstationary Navier-Stokes Problem. I. Regularity of solutions and second order error estimates for spatial discretization*, SIAM J. Numer. Anal. 19, 275–311 (1982).

[74] J. G. Heywood, R. Rannacher, and S. Turek: *Artificial boundary and flux and pressure conditions for the incompressible Navier-Stokes equations*, Int. J. Comput. Fluid Mecj. 22, 325–352 (1996).

[75] P. Hofmann: *Asymptotic expansions of thr discretization error of boundary value problems of the Laplace equation in rectngular domains*, Numer. Math. 9, 302–322 (1967).

[76] H. Huang, W. Han, and J. Zhou: *Extrapolation of numerical solutions for elliptic problems on corner domains*, Appl. Math. Comput. 83, 53-67 (1997).

[77] D. C. Joyce: *Survey of extrapolation processes in numerical analysis*, SIAM Review 13, 435–490 (1971).

[78] B. Lindberg: *Error estimation and iterative improvement for discretization algorithms*, BIT 20, 486–500 (1980).

[79] Q. Lin: *Fourth order eigenvalue approximation by extrapolation on domains with reentrant corners*, Numer. Math. 58, 631-640 (1991).

[80] Q. Lin and T. Lü: *Splitting extrapolations for multidimensional problems*, J. Comput. Math. 1, 45–51 (1983).

[81] Q. Lin and T. Lü: *The combination of approximate solutions for accelerating the convergence*, R.A.I.R.O. Anal. Numer. 12, 153–160 (1984).

[82] Q. Lin and T. Lü: *Asymtotic expansions for finite element approximation of elliptic problems on polygonal domains*, in Computing Methods in Applied Sciences and Engineering VI (R. Glowinski and J. L. Lions, eds), pp. 317–321, North-Holland. Amsterdam, 1984.

[83] Q. Lin and T. Lü: *Asymptotic expansions and extrapolation for the finite element method*, J. Systems Sci. Math. Sci. 5, 114-120 (1985) (in Chinese).

[84] Q. Lin, T. Lü, and S. M. Shen: *Maximum norm estimate, extrapolation and optimal points of stresses for the finite element method on the strongly regular triangulation*, J. Comput. Math. 1, 376–383 (1983).

[85] Q. Lin and R. F. Xie: *Some advances in the study of error expansion for finite elements, I. Eigenvalue error expansion*, J. Comput. Math. 4, 368-382 (1986).

[86] Q. Lin and R. F. Xie: *Error expansion for finite element approximation and its applications*, Proc. First China Conf. Numerical Methodsfor Partial Differential Equations, Shanghai, April 1987 (You-Lan Zhu, Ben-Yu Guo, eds.), Springer, 1987.

[87] Q. Lin and R. F. Xie: *How ro recover the convergence rate for Richardson extrapolation on bounded domains*, J. Comput. Math. 6, 81-92 (1988).

[88] Q. Lin and R. F. Xie: *Error expansion for FEM and superconvergence under natural assumptions*, J. Comput. Math. 7, 402-411 (1989).

[89] Q. Lin and J. C. Xu: *Linear finite elements with high accuracy*, J. Comput. Math. 3, 115-133 (1985).

[90] Q. Lin and Q. D. Zhu: *Asymptotic expansion for the derivative of finite elements*, J. Comput. Math. 2, 361-363 (1984).

[91] Q. Lin and Q. D. Zhu: *Local asymptotic expansions and extrapolation for finite elements*, J. Comput. Math. 4, 126–147 (1986).

[92] J. N. Lyness and A. C. Genz: *On simplex trapezoidal rule families*, SIAM J. Numer. Anal. 4, 263-265 (1986).

[93] G. Marchuk and V. Shaidurov: *Difference Methods and their Extrapolations*, Springer, Berlin, 1983.

[94] D. Meidner, R. Rannacher, and J. Vihharev: *Goal-oriented error control of the iterative solution of finite element equations*, J. Numer. Math. 17, 143-172 (2009).

[95] M. S. Mommer and R. Stevenson: *A goal-oriented adaptive finite element method with convergence rates*, SIAM J. Numer. Anal. 47, 861–886 (2009).

[96] H. Munz: *Uniform expansions for a class of finite difference schemes for elliptic boundary value problems*, Math. Comp. 36 155–170 (1981).

[97] P. Neittaanmäki and Q. Lin: *Acceleration of the convergence in finite difference method by predictor-corrector and splitting extrapolation methods*, J. Comput. Math. 5, 181–190 (1987).

[98] J. Nitsche: *Zur lokalen Konvergenz von Projektionen auf finite Elemente*, in "Approximation Theory", pp. 329–346, Lecture Notes in Math. 556, Springer, Heidelber-New York-Berlin, 1976.

[99] V. Pereyra, W. Proskurowski, and O. Widlund: *High ofrder fast Laplace solvers for the Dirichlet problem on general regions*, Math. Comp. 31, 1–16 (1977).

[100] R. Rannacher: *Some asymptotic error estimates for finite element approximation of minimal surfaces*, R.A.I.R.O. Anal. Numer. 11, 181-196 (1976).

[101] R. Rannacher: *On nonconforming and mixed finite element methods for plate bending problems. The linear case*, R.A.I.R.O. Anal. Numer. 13, 369-387 (1979).

[102] R. Rannacher: *On finite element approximation of general boundary value problems in nonlinear elasticity*, Calcolo 17, 175-193 (1980).

[103] R. Rannacher: *Richardson extrapolation with finite elements*, Proc. 2nd GAMM-Seminar "Numerical Techniques in Continuum Mechanics", Kiel 1986 (W. Hackbusch, K. Witsch, eds), pp. 90–101, Notes in Numer. Fluid Mech., Vieweg, Braunschweig, 1987.

[104] R. Rannacher: *Richardson extrapolation for a mixed finite element approximation of a plate bending problem*, Z. Angew. Math. Mech. 67, 381-383 (1987).

[105] R. Rannacher: *Extrapolation techniques in the finite element method: A survey*, in Proc. Summer School Numerical Analysis, Helsinki 1987, pp. 80113, 1988.

[106] R. Rannacher: *Defect correction techniques in the finite element method*, Proc. Conf. on Progress in Partial Differential Equations: The Metz Surveys (M.Chipot, J.Saint Jean Paulin, eds.), pp. 184-200, Longman, 1991.

[107] R. Rannacher: *On the convergence of the Newton-Raphson method for strongly nonlinear problems*, in Nonlinear Computational Mechanics, State of the Art (P. Wriggers and W. Wagner, eds), pp. 11–30, Springer, Berlin-Heidelberg-New York, 1991.

[108] R. Rannacher: *L^∞-stability estimates and asymptotic error expansions for parabolic finite element equations*, Proc. GAMM Seminar Extrapolations- und Defektkorrektur-Methoden, Heidelberg 1990 (J.Frehse, R. Rannacher, eds.), Bonn. Math. Schr. 228, 74-94 1992.

[109] R. Rannacher: *Finite element methods for the incompressible Navier-Stokes equations.* in Fundamental Directions in Mathematical Fluid Mechanics, Galdi GP, Heywood J and Rannacher R (eds). Birkhäuser: Basel-Boston-Berlin, 2000.

[110] R. Rannacher: *Incompressible viscous flow*, in Encyclopedia of Computational Mechanics (E. Stein, R. de Borst, T.J.R.Hughes, eds), Volume 3 'Fluids', John Wiley, Chichester, 2004.

[111] R. Rannacher: *A Short Course on numerical simulation of viscous flow: discretization, optimization and stability analysis*, in Lecture Notes 12th school "Mathematical Theory in Fluid Mechanics", Karcov, Czech Republic, Spring 2011, AIMS, Discrete and Continuous Dynamical Systems - Series S, Vol. 5(6), pp. 1147–1194, 2012.

[112] R. Rannacher: *Pointwise convergence of finite element approximations to quasi-nonlinear elliptic boundary value problems on non-quasi-uniform meshes*, Preprint, Heidelberg University, 2016.

[113] R. Rannacher and R. Scott: *Some optimal error estimates for piecewise linear finite element approximations*, Math. Comp. 31, 437-445 (1982).

[114] R. Rannacher and S. Turek: *Simple nonconforming quadrilateral Stokes element*, Numer. Meth. Part. Diff. Equ. 8, 97–111 (1992).

[115] R. Rannacher and J. Vihharev: *Adaptive finite element analysis of nonlinear problems: balancing of discretization and iteration errors*, J. Numer. Math. 21, 23–62 (2013).

[116] A. Westenberger, and W. Wollner: *Adaptive finite element approximation of eigenvalue problems: balancing discretization and iteration error*, J. Numer. Math. 18, 303–327 (2010).

[117] L. F. Richardson: *The approximate solution of physical problems involving differenbtial equations using finite differences, with an application to the stress in a masonry dam*, Phil. Trans. Roy. Soc. London A. 210, 307–357 (1911).

[118] L. F. Richardson: *The deferred approach to the limit, 1. The single lattice*, Phil. Trans. Roy. Soc. London A. 226, 299–349 (1927).

[119] Th. Richter: *A posteriori error estimation and anisotropy detection with the dual-weighted residual method*, Int. J. Num. Meth. Fluids 62, 90–118 (2010).

[120] Th. Richter and Th. Wick: *Variational Localizations of the Dual-Weighted Residual Estimator*, J. Comput. Appl. Math. (2014).

[121] A. H. Schatz: *Pointwise error estimates and asymptotic error expansion inequalities for the finite element method on irregular grids: Part I. Global estimates*, Math. Comp. 67 877-899 (1998).

[122] A. H. Schatz: *Pointwise error estimates and asymptotic error expansion inequalities for the finite element method in irregular grids: Part II. Interior estimates*, SIAM J. Numer. Anal. 38, 1269-1293 (2000).

[123] A. H. Schatz and L. B. Wahlbin: *Interior maximum norm estimates for finite element methods*, Math. Comp. 31, 414–442 (1977).

[124] A. H. Schatz and L. B. Wahlbin: *Maximum norm estimates in the finite element method on plane polygonal domains, Part I.*, Math. Comp. 32, 73–109 (1978).

[125] A. H. Schatz and L. B. Wahlbin: *Maximum norm estimates in the finite element method on plane polygonal domains, Part II. Refinements*, Math. Comp. 33, 465–492 (1979).

[126] R. Scholz: *A mixed method for 4th order problems using linear finite elements*, R.A.I.R.O. Anal. Numer. 12, 85–90 (1978).

[127] R. D. Skeel: *A theoretical framework for proving accuracy results for deferred corrections*, SIAM J. Numer. Anal. 19, 171–196 (1981).

[128] H. J. Stetter: *Asymptotic expansions for the error of discretization algorithms for non-linear functional equations*, Numer. Math. 7, 18–31 (1965).

[129] H. J. Stetter: *The defect correction principle and and discretization methods*, Numer. Math. 29, 425–443 (1978).

[130] R. Stevenson: *Optimality of a standard adaptive finite element method*, Found. Comput.Math. 7, 245–269 (2007).

[131] E. A. Volkov: *On a way of increasing the accuracy of the net method*, Soviet Math. Doklady 96, 685–688 (1954).

[132] E. A. Volkov: *A method for improving the accuracy if grid solutions of the Poisson equation*, Vycisl. Mat. 1, 62–80 (1957), in Russian, English translation: Amer. Math. Soc, Transl. 35, 117–136 (1964).

[133] L. P. Wang: *Superconvergence in Galerkin Finite Element Methods*, Springer, Berlin - Heidelberg, 1995.

[134] J. P. Wang: *Asymptotic expansions and L^∞-error estimates for mixed finite element methods for second order elliptic problems*, Numer. Math., 55, 401–430 (1989).

[135] W. Wasow: *Discrete approximations of elliptic differential equations*, Z. Angew. Math. Phys. 6, 81–97 (1966).

[136] W. Wasow: *Asymptotic Expansions for Ordinary Differential Equations*, John Wiley, 1965.

[137] C. Zenger and H. Gietl: *Improved difference schemes for the Dirichlet problem of Poisson's equation in the neighbourhood of corners*, Numer. Math. 30, 315–332 (1978).